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LITTLE BMO AND JOURNÉ COMMUTATORS

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Dedicated to Jean Esterle on the occasion of his 70-th birthday

ABSTRACT. We give a real variable extension to the parts of Ferguson-Sadosky's work that concern the characterization of the little BMO space on the polydisk via boundedness of commutators. Part of the extension we provide is contained in the recent article by the authors, where iterated commutators and more general mixed BMO classes are considered. In this note, we restrict ourselves to the simplest real variable case that allows a much simpler proof for the lower estimate of the commutator. We also include an improved upper estimate for commutators of Journé operators with multiplication of little BMO function.

1. INTRODUCTION

A classical result of Nehari [Ne57] says that a Hankel operator with antianalytic symbol b mapping analytic functions into the space of antianalytic functions by $f \mapsto P_- bf$ is bounded if and only if the symbol belongs to BMO. This theorem has an equivalent formulation by means of commutators of a symbol function b and the Hilbert transform $[H, b] = Hb - bH$, as the latter is a combination of orthogonal Hankel operators.

In [FS00] Ferguson and Sadosky investigate, among others, so-called big Hankel operators on the bidisk. Big Hankel operators are those who project onto the 'big' subspace of $L^2(\mathbb{T}^2)$ of functions antianalytic in either variable. They correspond to commutators with tensor products of Hilbert transforms $H_1 H_2 = H_1 \otimes H_2$

$$[H_1 H_2, b],$$

where $b = b(x_1, x_2)$ and H_i are the Hilbert transforms acting in the ith variable. Ferguson and Sadosky show that this commutator is bounded if and only if the symbol b belongs to the so called little BMO class, consisting of those functions that are uniformly in BMO in each variable separately. Their very clever proof makes use of Toeplitz operators. One deduces from their argument the two-sided estimate

$$\|b\|_{\mathrm{bmo}} \lesssim \|[H_1 H_2, b]\|_{2\to 2} \lesssim \|b\|_{\mathrm{bmo}},$$

where $\|\cdot\|_{\mathrm{bmo}}$ is the little BMO norm on the bidisk. The same result with more Hilbert transforms are implied by the method. In this note, we review the argument of Ferguson–Sadosky, however avoiding their notation using the Fourier transform. We also

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give a real variable argument that recovers their result on the Hilbert transform without the use of Toeplitz operators.

When leaving the notion of Hankel operators behind, their interpretation as commutators allow for natural generalizations. The classical text [CRW76] of Coifman, Rochberg and Weiss extended the one-parameter theory to real analysis in the sense that the Hilbert transforms were replaced by Riesz transforms. In their text, they obtained sufficiency, i.e. that a BMO symbol b yields an L^2 bounded commutator for certain more general, convolution type singular integral operators. For necessity, they showed that the collection of Riesz transforms was representative enough. One deduces the two-sided estimate

$$\|b\|_{\mathrm{BMO}} \lesssim \sup_j \|[R_j, b]\|_{2\to 2} \lesssim \|b\|_{\mathrm{BMO}},$$

where $\|\cdot\|_{\mathrm{BMO}}$ is the usual one-parameter BMO norm in $\mathbb{R}^n$. In this note, we prove that

$$\|b\|_{\mathrm{bmo}} \lesssim \sup_{i,j} \|[R_i^1 R_j^2, b]\|_{2\to 2} \lesssim \|b\|_{\mathrm{bmo}}, \tag{1.1}$$

where $b = b(x_1, x_2)$ with $x_{1,2} \in \mathbb{R}^n$ and $R_*^{1,2}$ Riesz transforms in $\mathbb{R}^n$. The same result with tensor products of higher orders is implicit and presents no additional difficulties.

In their recent text [OPS16], the authors were interested in commutators of the form $[H^1, [H^2 H^3, b]]$ or $[R^1, [R^2 R^3, b]]$. The argument of Ferguson-Sadosky could be recovered for the Hilbert transform case (using a deep result [FL02] of Ferguson-Lacey as a base) but there is a sharp contrast to the real variable case, whose lower estimate turned out substantially more difficult. In this situation here, where no iterations are present in the commutator, an easier argument can be used to handle the Riesz transform case, since the lower estimate in (1.1) can be obtained by directly estimating the little BMO norm of the symbol.

The upper estimate in (1.1) follows easily from the upper estimate of Coifman-Rochberg-Weiss. However, the upper estimate is also true when T is a paraproduct-free multi-parameter Journé operator (see [OPS16]):

$$\|[T, b]\|_{2\to 2} \lesssim \|b\|_{\mathrm{BMO}}, \tag{1.2}$$

with constants depending upon the defining constants of the operator T. Its proof is very different from that in [CRW76] in that it uses the notion of Haar shift in a very general form. The use of Haar shifts for commutator estimates originated in [Pe00] for a certain commutator involving the Hilbert transform. It has since been used for upper estimates in [LPPW10], [DO16], [HLW16], [HLW] in a much more general setting, in part using Hytonen's representation formula [Hy12].

The estimate (1.2) (and its more general iterated analogs) are contained in the recent paper by the authors [OPS16] and uses the representation formulae for Journé operators by Martikainen [Ma12] and Ou [Ou]. We improve this estimate in this note by weakening the paraproduct-free condition assumed on T and include a self contained proof for the bi-parameter case, with the multi-parameter case being an implicit generalization.

1.1. H^1 **and BMO.** The (real) Hardy space $H^1(\mathbb{R}^n)$ typically denotes the class of functions with the norm

$$\|f\|_{H^1(\mathbb{R}^n)} = \sum_{j=0}^{n} \|R_j f\|_1$$

where R_j denotes the jth Riesz transform. Here and below we adopt the convention that R_0, the 0th Riesz transform, is the identity. This space is invariant under the one parameter family of isotropic dilations. It is well known that the dual of this space is BMO, the space of functions of bounded mean oscillation.

$$\|b\|_{\mathrm{BMO}} \quad = \quad \sup_Q \frac{1}{|Q|} \int_Q |b(x) - \langle b\rangle_Q| dx$$

where $\langle b\rangle_Q = |Q|^{-1} \int_Q b$, the average of b over the cube Q.

1.2. **Little bmo.** Writing $\mathbb{R}^{2n}$ for $(\mathbb{R}^n)^2$, a function $b \in L^1_{\mathrm{loc}}(\mathbb{R}^{2n})$ is in bmo$(\mathbb{R}^{2n})$ if and only if

$$\|b\|_{\mathrm{bmo}} = \sup_{Q_1\times Q_2} |Q_1|^{-1}|Q_2|^{-1} \int_{Q_1\times Q_2} |b(x_1,x_2) - \langle b\rangle_{Q_1\times Q_2}| dx_1 dx_2$$

with cubes Q_1, Q_2 in $\mathbb{R}^n$. It is easy to see that

$$\|b\|_{\mathrm{bmo}} \sim \max\{\sup_{x_1} \|b(x_1,\cdot)\|_{\mathrm{BMO}}; \sup_{x_2} \|b(\cdot,x_2)\|_{\mathrm{BMO}}\}.$$

The methods used in this text do not change if we treat more than two parameters, for which reason we restrict our exposition to this case. However, here is the definition of little BMO in the general case. A function $b \in L^1_{\mathrm{loc}}(\mathbb{R}^{\vec{d}})$ is in bmo$(\mathbb{R}^{\vec{d}})$ if and only if

$$\|b\|_{\mathrm{bmo}} = \sup_{\vec{Q}=Q_1\times\cdots\times Q_t} |\vec{Q}|^{-1} \int_{\vec{Q}} |b(\vec{x}) - \langle b\rangle_{\vec{Q}}| < \infty.$$

Here the Q_s are d-dimensional cubes and $\langle b\rangle_{\vec{Q}}$ denotes the average of b over $\vec{Q}$. It is easy to see that this space coincides with functions b that are uniformly in BMO in each variable separately. Let $\overrightarrow{x_{\hat{k}}} = (x_1, x_2, \ldots, x_{k-1}, \cdot, \ldots, x_t)$.

$$\|b(\overrightarrow{x_{\hat{k}}})\|_* = \sup_{Q_k} |Q_k|^{-1} \int_{Q_k} |b(x_k) - \langle b(\overrightarrow{k_{\hat{k}}})\rangle_{Q_k}|$$

with $\|b\|_{\mathrm{bmo}} \sim \max_k \left\{\sup_{\overrightarrow{x_{\hat{k}}}} \|b(\overrightarrow{x_{\hat{k}}})\|_*\right\}$. The predual of this space consists of functions f that can be written as $f = f_1 + f_2 + \ldots + f_t$ where f_k is such that $\sum_{j=0}^{d} \|R_j f_k\|_1 < \infty$.

Similar definitions hold in the Hilbert transform case as well as on the disk or torus.

2. The Hilbert transform case of Ferguson-Sadosky

The original result in [FS00] is stated on the polydisk. For this reason we present their argument in this setting with the obvious modifications of the definitions above. The proof also works in $\mathbb{R}^2$.

Theorem 2.1 (Ferguson–Sadosky). *There is the equivalence of norms*

$$\|b\|_{bmo} \lesssim \|[H_1 H_2, b]\|_{2\to 2} \lesssim \|b\|_{bmo}.$$

Proof. It is shown that for $b \in L^1(\mathbb{T}^2)$ the following are equivalent with linear relations of their norms:

(1) $b \in \text{bmo}$.
(2) The commutators $[H_1, b]$ and $[H_2, b]$ are bounded on $L^2(\mathbb{T}^2)$.
(3) The commutator $[H_2 H_1, b]$ is bounded on $L^2(\mathbb{T}^2)$.

We denote by P_i the projection onto the non-negative Fourier coefficients in variable i and by $P_i^\perp$ its orthogonal compliment in that variable. First observe that the commutators $[H_1, b]$ and $[H_2, b]$ are bounded on $L^2(\mathbb{T}^2)$ if and only if all four operators $P_i b P_i^\perp, P_i^\perp b P_i$ with $i \in \{1,2\}$ are bounded on $L^2(\mathbb{T}^2)$. The commutator $[H_2 H_1, b]$ is bounded on $L^2(\mathbb{T}^2)$ if and only if the two operators $Q_{12} b Q_{12}^\perp, Q_{12}^\perp b Q_{12}$ with $Q_{12} = P_1 P_2 + P_1^\perp P_2^\perp$ and $Q_{12}^\perp = P_1^\perp P_2 + P_1 P_2^\perp$, are bounded on $L^2(\mathbb{T}^3)$. Indeed, we have for $i = 1,2$

$$[H_1 H_2, b] = 2(Q_{12}^\perp b Q_{12} - Q_{12} b Q_{12}^\perp)$$

and

$$[H_i, b] = 2(P_i b P_i^\perp - P_i^\perp b P_i).$$

These can be seen by writing $H = P - P^\perp$ and by inserting $I = P + P^\perp$ where no Hilbert transform acts. (Observe that we omit a multiplicative constant of the Hilbert transform). Notice that the ranges of all arising summands are mutually orthogonal, so these operators are bounded if and only if each one of their summands is. So, typical terms that arise in (2) are of the form $P_1 b P_1^\perp$ and in (3) of the form $P_2 P_1 b P_1^\perp P_2$. The latter is a Toeplitz operator with symbol $P_1 b P_1^\perp$. The L^∞ norm of the collection of such symbols is comparable to $\|b\|_{\text{bmo}}$. □

Below is an alternative, direct proof of the lower estimate of

$$\|b\|_{\text{bmo}} \lesssim \|[b, H_1 H_2]\|_{2\to 2} \lesssim \|b\|_{\text{bmo}}$$

on the real line without the use of operator theory. Both upper and lower inequalities extend to the Riesz transform case and are written below. The upper estimate in the Hilbert transform case is exactly the same as the one written below at the beginning of section 3. The latter can be avoided by a well known argument using dyadic shifts. (See [Pe00] as well as the part of this note on the upper estimates). We present the proof for the lower bound in this special case:

Proof. Let $\Gamma_Q(x, y) = \text{sign}(b(x, y) - \langle b\rangle_Q)\mathbf{1}_Q(x, y)$. Now we rewrite the integrands appearing in the definition of little BMO:

$$\begin{aligned} &|Q||b(x,y) - \langle b\rangle_Q|\mathbf{1}_Q(x,y) \\ =&|Q|(b(x,y) - \langle b\rangle_Q)\Gamma_Q(x,y) \\ =&\int_Q \left(\frac{b(x,y) - b(x',y')}{(x-x')(y-y')}\right)(x-x')(y-y')\Gamma_Q(x,y)\,dx'dy' \\ =&[b, H_1 H_2]((x-x')(y-y')\mathbf{1}_Q(x',y'))\Gamma_Q(x,y). \end{aligned}$$

The commutator acts in the variables x' and y' and in the above notation, the x and y become multiplicative constants. Integrating a $|Q|^{-2}$ multiple of this equality in $dxdy$

over Q gives us on the left hand side the little BMO expression and on the right hand side

$$\begin{aligned}
&|Q|^{-2}\int_{\mathbb{R}^2}[b,H_1H_2]((x-x')(y-y')\mathbf{1}_Q(x',y'))\Gamma_Q(x,y)dxdy\\
\leq&|Q|^{-2}\|[b,H_1H_2]((x-x')(y-y')\mathbf{1}_Q(x',y'))\mathbf{1}_Q(x,y)\|_2\|\Gamma_Q(x,y)\|_2\\
\leq&|Q|^{-2}\|[b,H_1H_2]\|_{2\to 2}\|(x-x')(y-y')\mathbf{1}_Q(x',y')\mathbf{1}_Q(x,y)\|_2\|\Gamma_Q(x,y)\|_2.
\end{aligned}$$

Assuming for a moment that Q is centered about the origin, we obtain due to the supports of Γ_Q and $\mathbf{1}_Q$ the estimate

$$\|(x-x')(y-y')\mathbf{1}_Q(x',y')\mathbf{1}_Q(x,y)\|_2\|\Gamma_Q(x,y)\|_2 \lesssim |Q|^{3/2}|Q|^{1/2}=|Q|^2.$$

A translation argument on b finishes the proof. □

3. THE RIESZ TRANSFORM CASE

In this section we show the real variable analog of a theorem by Ferguson and Sadosky. We show that

$$\|b\|_{\mathrm{bmo}} \lesssim \sup_{i,j}\|[R_i^1R_j^2,b]\|_{2\to 2}\lesssim \|b\|_{\mathrm{bmo}}$$

where R_i^s are the Riesz transforms acting in the s^{th} variable.

The upper estimate is easy to deduce from [CRW76] or [LPPW10] thanks to the tensor product structure: let R^k denote any Riesz transform acting in the kth variable. Then

$$[R^1R^2,b]=[R^1\otimes R^2,b]=[(R^1\otimes I)\circ(I\otimes R^2),b].$$

Using that $[A\circ B,b]=A\circ[B,b]+[A,b]\circ B$, we see that

$$[R^1R^2,b]=R^1[R^2,b]+[R^1,b]R^2. \tag{3.1}$$

Consider $f=f(x_1)$ and $g=g(x_2)$. Observe that $[R^1,b](fg)=g\cdot[R^1,b](f)$, and so $\|[R^1,b](fg)\|^2_{L^2(\mathbb{R}^{2n})}=\|Fg\|^2_{L^2(\mathbb{R}^n)}$ where $F(x_2)=\|[R^1,b](f)\|_{L^2(\mathbb{R}^n)}$. The map $g\mapsto Fg$ has $L^2(\mathbb{R}^n)$ operator norm $\|F\|_\infty$. Thanks to the upper estimate in [CRW76],

$$\sup_{x_2}\|[R^1,b](f)\|_{L^2(\mathbb{R}^n)}\lesssim \sup_{x_2}\|b(\cdot,x_2)\|_{\mathrm{BMO}}\|f\|_2\lesssim\|b\|_{\mathrm{bmo}}\|f\|_2.$$

By reversing the role of the variables, using that Riesz transforms are bounded operators and the equality (3.1), the desired estimate for $\|[R^1,b]\|_{L^2(\mathbb{R}^{2n})\to L^2(\mathbb{R}^{2n})}$ follows.

To see the lower bound, one develops an argument found in [CRW76]. It is a generalisation of the direct proof of the lower bound in the Hilbert transform case. Let $\{X_k(x)\}$ and $\{Y_l(y)\}$ both be bases for the space of spherical harmonics of degree n in $\mathbb{R}^n$ respectively. Then $\sum_k|X_k(x)|^2=c_n|x|^{2n}$ (see [CW71] p.144) and thus

$$1=\frac{1}{c_n}\sum_k\frac{X_k(x-x')}{|x-x'|^{2n}}X_k(x-x').$$

Furthermore $X_k(x-x')=\sum_{|\alpha|+|\beta|=n}\mathbf{x}^{(k)}_{\alpha\beta}x^\alpha x'^\beta$ and equally for Y_l. Remember that

$$b(x,y)\in\mathrm{bmo}\iff\|b\|_{\mathrm{bmo}}=\sup_Q\frac{1}{|Q|}\int_Q|b(x,y)-\langle b\rangle_Q|dxdy<\infty.$$

Here, $Q = I \times J$ and I and J are cubes in $\mathbb{R}^n$. Let us define the function

$$\Gamma_Q(x,y) = \operatorname{sign}(b(x,y) - \langle b\rangle_Q)\mathbf{1}_Q(x,y).$$

So

$$\begin{aligned}
&|b(x,y)-\langle b\rangle_Q||Q|\mathbf{1}_Q(x,y) = (b(x,y)-\langle b\rangle_Q)|Q|\Gamma_Q(x,y)\\
&=\int_Q (b(x,y)-b(x',y'))\Gamma_Q(x,y)dx'dy'\\
&\sim\sum_{k,l}\int_Q (b(x,y)-b(x',y'))\frac{X_k^2(x-x')}{|x-x'|^{2n}}\frac{Y_l^2(y-y')}{|y-y'|^{2n}}\Gamma_Q(x,y)dx'dy'\\
&=\sum_{k,l}\int_{\mathbb{R}^{2n}}\frac{b(x,y)-b(x',y')}{|x-x'|^{2n}|y-y'|^{2n}}X_k(x-x')Y_l(y-y')\cdot\\
&\quad\cdot\sum_{|\alpha|+|\beta|=n}\mathbf{x}_{\alpha\beta}^{(k)}x^\alpha x'^\beta\sum_{|\gamma|+|\delta|=n}\mathbf{y}_{\gamma\delta}^{(l)}y^\gamma y'^\delta\Gamma_Q(x,y)\mathbf{1}_Q(x',y')dx'dy',
\end{aligned}$$

where the last line comes from the development of X_kY_l in the standard basis of polynomials. Note that

$$\begin{aligned}
&\int_{\mathbb{R}^{2n}}\frac{b(x,y)-b(x',y')}{|x-x'|^{2n}|y-y'|^{2n}}X_k(x-x')Y_l(y-y')x'^\beta y'^\delta\mathbf{1}_Q(x',y')dx'dy'\\
&=[b,T_kT_l](x'^\beta y'^\delta\mathbf{1}_Q(x',y')).
\end{aligned}$$

Here $T_k$and T_l are the Calderón-Zygmund operators that correspond to the kernels

$$\frac{X_k(x)}{|x|^{2n}} \text{ and } \frac{Y_l(y)}{|y|^{2n}}.$$

Observe that these have the correct homogeneity due to the homogeneity of the X_k and Y_l. With this notation, the above becomes

$$\begin{aligned}
&|b(x,y)-\langle b\rangle_Q||Q|\mathbf{1}_Q(x,y)\\
&=\sum_{k,l}\sum_{|\alpha|+|\beta|=n}\sum_{|\gamma|+|\delta|=n}\mathbf{x}_{\alpha\beta}^{(k)}x^\alpha\mathbf{y}_{\gamma\delta}^{(l)}y^\gamma\Gamma_Q(x,y)[b,T_kT_l](x'^\beta y'^\delta\mathbf{1}_Q(x',y'))(x,y).
\end{aligned}$$

Now, we integrate with respect to (x,y) to obtain via Cauchy Schwarz

$$\begin{aligned}
&|Q|\int_Q|b(x,y)-\langle b\rangle_Q|dxdy\\
&\lesssim\sum_{k,l}\sum_{|\alpha|+|\beta|=n}\sum_{|\gamma|+|\delta|=n}\|\mathbf{x}_{\alpha\beta}^{(k)}x^\alpha\mathbf{y}_{\gamma\delta}^{(l)}y^\gamma\Gamma_Q(x,y)\|_2\|[b,T_kT_l](x'^\beta y'^\delta\mathbf{1}_Q(x',y'))\|_2\\
&\lesssim\sum_{k,l}\sum_{|\alpha|+|\beta|=n}\sum_{|\gamma|+|\delta|=n}\|x^\alpha y^\gamma\Gamma_Q(x,y)\|_2\|[b,T_kT_l]\|_{2\to2}\|x'^\beta y'^\delta\mathbf{1}_Q(x',y')\|_2.
\end{aligned}$$

In the last line, we deleted the coefficients of the X and Y at the cost of a constant. Now let us assume for a moment that both I and J are centered at 0 and thus Q centered at 0. In this case, since Γ_Q and $\mathbf{1}_Q$ are supported in Q, there is only contribution for x, x', y, y'

in Q.

$$\cdots \lesssim \sum_{k,l} \mathfrak{l}(I)^{|\alpha|}\mathfrak{l}(I)^{|\beta|}\mathfrak{l}(J)^{|\gamma|}\mathfrak{l}(J)^{|\delta|}|Q|^{1/2+1/2}\|[b,T_kT_l]\|_{2\to 2}$$
$$\lesssim \sum_{k,l} |Q|^2\|[b,T_kT_l]\|_{2\to 2}.$$

Notice that the T_k and T_l are homogeneous polynomials in Riesz transforms. Therefore the commutator $[b,T_kT_l]$ can be written as a linear combination of terms of the form $M[b,R_i^1R_j^2]N$ where M and N are compositions of Riesz transforms: in a first step write $[b,T_kT_l]$ as linear combination of terms of the form $[b,R^k_{(n)}R^l_{(n)}]$ where

$$R^k_{(n)} = \prod_s R^1_{i_s^{(k)}}$$

is a composition of n Riesz transforms acting in the variabe 1 with a choice $i^{(k)} = (i_s^{(k)})_{s=1}^n \in \{1,\dots,n\}^n$ for each k and similar for $R^l_{(n)}$ acting in variable 2. Then, for each term, apply $[AB,b] = A[B,b]+[A,b]B$ successively as follows. Use $A = R^1_{i_1}R^2_{j_1}$ and B of the form $R^k_{(n-1)}R^l_{(n-1)}$ and repeat. It then follows that for each k,l the commutator $[b,T_kT_l]$ writes as a linear combination of terms such as $M[b,R_i^1R_j^2]N$ where M and N are compositions of Riesz transforms. It is decisive that T_k and T_l are homogeneous polynomials in Riesz transforms of the same degree. We required that all commutators of the form $[b,R_i^1R_j^2]$ are bounded, we have shown the bmo estimate for b for rectangles Q whose sides are centered at 0. We now translate b in the two directions separately and obtain what we need.

4. Sufficiency cases in the Journé setting

The class of bi-parameter singular integral operators treated in this section is that of a Journé type operator [Jou85] satisfying a certain weak boundedness property, which we define as follows:

Definition 4.1. A continuous linear mapping

$$T : C_0^\infty(\mathbb{R}^n)\otimes C_0^\infty(\mathbb{R}^m) \to [C_0^\infty(\mathbb{R}^n)\otimes C_0^\infty(\mathbb{R}^m)]'$$

is called a *(paraproduct-free) bi-parameter Calderón-Zygmund operator* if the following conditions are satisfied:

(1) T is a Journé type bi-parameter δ-singular integral operator, i.e. there exists a pair (K_1,K_2) of δCZ-δ-standard kernels so that, for all $f_1,g_1\in C_0^\infty(\mathbb{R}^n)$ and $f_2,g_2\in C_0^\infty(\mathbb{R}^m)$,

$$\langle T(f_1\otimes f_2), g_1\otimes g_2\rangle = \int f_1(y_1)\langle K_1(x_1,y_1)f_2,g_2\rangle g_1(x_1)\,dx_1dy_1$$

when $\operatorname{spt} f_1\cap \operatorname{spt} g_1 = \emptyset$;

$$\langle T(f_1\otimes f_2), g_1\otimes g_2\rangle = \int f_2(y_2)\langle K_2(x_2,y_2)f_1,g_1\rangle g_2(x_2)\,dx_2dy_2$$

when $\operatorname{spt} f_2\cap \operatorname{spt} g_2 = \emptyset$.

(2) T satisfies the weak boundedness property

$$|\langle T(\mathbf{1}_I\otimes \mathbf{1}_J), \mathbf{1}_I\otimes \mathbf{1}_J\rangle| \lesssim |I||J|,$$

for any cubes $I\subset \mathbb{R}^n, J\in\mathbb{R}^m$.

(3) T is paraproduct free in the sense that

$$T(1\otimes\cdot)=T(\cdot\otimes 1)=T^*(1\otimes\cdot)=T^*(\cdot\otimes 1)=0.$$

Recall that a δCZ-δ-standard kernel is a vector valued standard kernel taking values in the Banach space consisting of all Calderón-Zygmund operators.

Martikainen in [Ma12] proved a representation theorem saying that any bi-parameter Journé operator can be represented as average of bi-parameter dyadic shifts and paraproducts. The multi-parameter analog is found in [Ou]. In a recent paper by the authors, it is shown that when T in this class is paraproduct-free, there holds the following estimate:

$$\|[b,T]\|_{L^2\to L^2}\lesssim \|b\|_{\mathrm{bmo}}, \tag{4.1}$$

where the underlying constant depends only on the dimension and $\|T\|_{CZ}$. The structure of the proof is the following: 1. implied by Martikainen's representation theorem, it suffices to prove the boundedness of $[b,S]$, where S is any dyadic shift or paraproduct; 2. as T is paraproduct-free, in the representation, all the paraproduct terms are equal to zero, hence one ends up with only dyadic shifts; 3. the boundedness of such $[b,S]$ can be obtained by representing it as linear combinations of bounded paraproducts.

In this note, we would like to prove (4.1) (referred to as the *upper bound*) for more general bi-parameter Journé operators which are not necessarily paraproduct-free, *via* a study of commutators of bi-parameter paraproducts. According to Martikainen's representation theorem, the gap between paraproduct-free operators and the general Journé's operators lies in the presence of three types of bi-parameter dyadic paraproducts (modulo symmetry) defined as follows:

Recalling that h_I and h_I^1 denote cancellative and non-cancellative Haar functions, respectively, the standard paraproduct

$$\Pi_1(f):=\sum_{I_1,I_2} a_{I_1I_2}\langle f,h_{I_1}\otimes h_{I_2}\rangle h^1_{I_1}\otimes h^1_{I_2},$$

the mixed paraproduct

$$\Pi_2(f):=\sum_{I_1,I_2} a_{I_1,I_2}\langle f,h_{I_1}\otimes h^1_{I_2}\rangle h^1_{I_1}\otimes h_{I_2},$$

and the partial paraproduct for any fixed $i,j\geq 0$

$$S^{ij}(f):=\sum_{K_1}\sum^{(i,j)}_{I_1,J_1\subset K_1}\sum_{K_2} a_{I_1J_1K_1K_2}\langle f,h_{I_1}\otimes h_{K_2}\rangle h_{J_1}\otimes h^1_{K_2},$$

where in the first two operators, $a_{I_1I_2}=\langle a,h_{I_1}\otimes h_{I_2}\rangle |I_1|^{-1/2}|I_2|^{-1/2}$ for some fixed product BMO symbol function a such that $\|a\|_{\mathrm{BMO}}\leq 1$, which also implies that $|a_{I_1I_2}|\leq 1$. And in the last operator, the notation $\sum^{(i,j)}_{I_1,J_1\subset K_1}$ denotes the sum over all the sub cubes I_1,J_1 contained in K_1 such that $\ell(I_1)=2^{-i}\ell(K_1), \ell(J_1)=2^{-j}\ell(K_1)$, and for any fixed I_1,J_1,K_1, there exists a BMO function $a_{I_1J_1K_1}$ in the second variable such that $a_{I_1J_1K_1K_2}=\langle a_{I_1J_1K_1},h_{K_2}\rangle_2|K_2|^{-1/2}$ and $\|a_{I_1J_1K_1}\|_{\mathrm{BMO}}\leq \frac{|I_1|^{1/2}|J_1|^{1/2}}{|K_1|}$.

In the following, we will study the upper bound for commutators of the standard paraproduct Π_1 and prove the following theorem:

Theorem 4.2. *Let Π_1 be a standard bi-parameter dyadic paraproduct:*

$$\Pi_1(f) := \sum_{I_1,I_2} a_{I_1 I_2}\langle f, h_{I_1}\otimes h_{I_2}\rangle h^1_{I_1}\otimes h^1_{I_2},$$

where $a_{I_1 I_2} := \langle a, h_{I_1}\otimes h_{I_2}\rangle |I_1|^{-1/2}|I_2|^{-1/2}$ for some product BMO symbol a satisfying $\|a\|_{BMO}\leq 1$. Then for any little BMO function b,

$$\|[b,\Pi_1]f\|_{L^2(\mathbb{R}^n\times\mathbb{R}^m)} \lesssim \|b\|_{bmo}\|f\|_{L^2(\mathbb{R}^n\times\mathbb{R}^m)}, \quad \forall f\in L^2(\mathbb{R}^n\times\mathbb{R}^m).$$

This implies directly that for any bi-parameter Journé operator that are mixed paraproduct and partial paraproduct free, there holds the upper estimate (4.1). Before proceeding with the proof, we'd like to address the fact that (4.1) holds true whenever T is a tensor product of two one-parameter Calderón-Zygmund operators, which is a direct consequence of the main theorem in [DO16]. Therefore, a key difficulty we are facing here is the lack of tensor product structure of the symbol functions of the bi-parameter paraproducts, which is also closely related with the subtlety of the product BMO spaces.

We adopt the by now standard technique: fix first an L^2 function f, decompose $[b,\Pi_1]f$ using Haar bases, then represent the sum as finite linear combinations of basic bounded operators such as paraproducts.

Decompose

$$[b,\Pi_1]f = \sum_{I_1,I_2}\sum_{J_1,J_2}\langle b, h_{I_1}\otimes h_{I_2}\rangle\langle f, h_{J_1}\otimes h_{J_2}\rangle [h_{I_1}\otimes h_{I_2},\Pi_1]h_{J_1}\otimes h_{J_2}.$$

By the definition of Π_1 and the cancellation of the commutator, it is not hard to observe that $[h_{I_1}\otimes h_{I_2},\Pi_1]h_{J_1}\otimes h_{J_2}$ is nonzero only if $I_1\subset J_1$ or $I_2\subset J_2$. (We refer the readers to [OPS16] for more details.) The part $I_1\subset J_1, I_2\subset J_2$ is called the regular case, while the part $I_1\subset J_1, I_2\supsetneq J_2$ and the symmetric one are called the mixed cases. Moreover, we will deal with the sum corresponding to the first term $h_{I_1}\otimes h_{I_2}\Pi_1(h_{J_1}\otimes h_{J_2})$ and the second term $\Pi_1(h_{I_1}h_{J_1}\otimes h_{I_2}h_{J_2})$ separately.

4.1. **Second term.** The estimate of the sum corresponding to the second term is usually much easier, due to the fact that the operator Π_1 can be pulled outside of the sum by linearity. Since Π_1 is bounded on L^2, it suffices to prove that

$$\Big\|\sum_{I_1,I_2}\sum_{J_1,J_2}\langle b, h_{I_1}\otimes h_{I_2}\rangle\langle f, h_{J_1}\otimes h_{J_2}\rangle h_{I_1}h_{J_1}\otimes h_{I_2}h_{J_2}\Big\|_{L^2}\lesssim \|b\|_{\text{bmo}}\|f\|_{L^2}.$$

In the regular case when $I_1\subset J_1, I_2\subset J_2$, split

$$\sum_{I_1\subset J_1}\sum_{I_2\subset J_2} = \sum_{I_1\subsetneq J_1}\sum_{I_2\subsetneq J_2} + \sum_{I_1\subsetneq J_1}\sum_{I_2=J_2} + \sum_{I_1=J_1}\sum_{I_2\subsetneq J_2} + \sum_{I_1=J_1}\sum_{I_2=J_2} =: I+II+III+IV,$$

then each of the four parts could be represented as a classical dyadic paraproduct, whose boundedness in terms of the product BMO norm of b (which is less than $\|b\|_{\text{bmo}}$) is well known. For example

$$I = \sum_{I_1,I_2}\langle b, h_{I_1}\otimes h_{I_2}\rangle\langle f, h^1_{I_1}\otimes h^1_{I_2}\rangle h_{I_1}\otimes h_{I_2}|I_1|^{-1/2}|I_2|^{-1/2},$$

$$IV = \sum_{I_1, I_2} \langle b, h_{I_1} \otimes h_{I_2} \rangle \langle f, h_{I_1} \otimes h_{I_2} \rangle h_{I_1}^{\epsilon_1} \otimes h_{I_2}^{\epsilon_2} |I_1|^{-1/2} |I_2|^{-1/2},$$

and similarly for part II and III.

In the mixed case when $I_1 \subset J_1, I_2 \supsetneq J_2$, we have to exploit the full strength of the little BMO norm. Summing over I_2 first gives

$$\begin{aligned}
&\sum_{I_1 \subset J_1} \sum_{I_2 \supsetneq J_2} \langle b, h_{I_1} \otimes h_{I_2} \rangle \langle f, h_{J_1} \otimes h_{J_2} \rangle h_{I_1} h_{J_1} \otimes h_{I_2} h_{J_2} \\
&= \sum_{I_1 \subset J_1} \sum_{J_2} \langle b, h_{I_1} \otimes h^1_{J_2} \rangle \langle f, h_{J_1} \otimes h_{J_2} \rangle h_{I_1} h_{J_1} \otimes h_{J_2} |J_2|^{-1/2} \\
&= \sum_{J_2} \Big(\sum_{I_1 \subsetneq J_1} + \sum_{I_1 = J_1} \Big).
\end{aligned}$$

In the above, observe that

$$\sum_{J_2} \sum_{I_1 \subsetneq J_1} = \sum_{I_1, J_2} \langle b, h_{I_1} \otimes h^1_{J_2} \rangle \langle f, h^1_{I_1} \otimes h_{J_2} \rangle h_{I_1} \otimes h_{J_2} |I_1|^{-1/2} |J_2|^{-1/2},$$

and

$$\sum_{J_2} \sum_{I_1 = J_1} = \sum_{I_1, J_2} \langle b, h_{I_1} \otimes h^1_{J_2} \rangle \langle f, h_{I_1} \otimes h_{J_2} \rangle h^{\epsilon}_{I_1} \otimes h_{J_2} |I_1|^{-1/2} |I_2|^{-1/2},$$

where ϵ could be equal to $\vec{1}$. Then both of them are paraproduct with little BMO symbol, whose boundedness follows from the following lemma.

Lemma 4.3. *Let b be a little BMO function on $\mathbb{R}^n \times \mathbb{R}^m$. For paraproducts of the following two types:*

$$B^1_{k,l}(b, f) := \sum_{I_1, I_2} \beta_{I_1, I_2} \langle b, h^1_{I_1^{(k)}} \otimes h_{I_2^{(l)}} \rangle \langle f, h_{I_1} \otimes h^{\epsilon}_{I_2} \rangle h_{I_1} \otimes h^{\epsilon'}_{I_2} |I_1^{(k)}|^{-1/2} |I_2^{(l)}|^{-1/2},$$

$$B^2_{k,l}(b, f) := \sum_{I_1, I_2} \beta_{I_1, I_2} \langle b, h_{I_1^{(k)}} \otimes h^1_{I_2^{(l)}} \rangle \langle f, h^{\epsilon}_{I_1} \otimes h_{I_2} \rangle h^{\epsilon'}_{I_1} \otimes h_{I_2} |I_1^{(k)}|^{-1/2} |I_2^{(l)}|^{-1/2},$$

where at least one of ϵ, ϵ' is not equal to $\vec{1}$ and $|\beta_{I_1, I_2}| \le 1$ uniformly, there holds

$$\|B^i_{k,l}(b, f)\|_{L^2} \lesssim \|b\|_{bmo} \|f\|_{L^2}, \quad i = 1, 2.$$

Proof. We only prove the result for $B^1_{k,l}$ as the other one is completely symmetric. Write

$$\begin{aligned}
B^1_{k,l}(b, f) &= \sum_{I_1} h_{I_1} \otimes \left(\sum_{I_2} \beta_{I_1, I_2} \langle \langle b \rangle_{I_1^{(k)}}, h_{I_2^{(l)}} \rangle_2 \langle \langle f, h_{I_1} \rangle_1, h^{\epsilon}_{I_2} \rangle_2 h^{\epsilon'}_{I_2} |I_2^{(l)}|^{-1/2} \right) \\
&= \sum_{I_1} h_{I_1} \otimes B^2_l (\langle b \rangle_{I_1^{(k)}}, \langle f, h_{I_1} \rangle_1),
\end{aligned}$$

where B^2_l is a one-parameter paraproduct with descendents in the second variable. Then,

$$\begin{aligned}
\|B^1_{k,l}(b, f)\|^2_{L^2} &= \sum_{I_1} \|B^2_l (\langle b \rangle_{I_1^{(k)}}, \langle f, h_{I_1} \rangle_1)\|^2_{L^2(\mathbb{R}^m)} \\
&\lesssim \sum_{I_1} \|\langle b \rangle_{I_1^{(k)}}\|^2_{BMO(\mathbb{R}^m)} \|\langle f, h_{I_1} \rangle_1\|^2_{L^2(\mathbb{R}^m)} \\
&\le \|b\|^2_{\mathrm{bmo}} \sum_{I_1} \|\langle f, h_{I_1} \rangle_1\|^2_{L^2(\mathbb{R}^m)} = \|b\|^2_{\mathrm{bmo}} \|f\|^2_{L^2}.
\end{aligned}$$

□

4.2. **First term.** In order to deal with the sum corresponding to the first term, other than paraproduct, one needs another type of basic operators defined as

$$\begin{aligned}&\tilde{PB}_l(a,\{b_{I_2}\}_{I_2\in\mathscr{D}^m},f)\\&:=\sum_{I_1,I_2}\langle a,h_{I_1}\otimes h_{I_2^{(l)}}\rangle\langle f,h_{I_1}\otimes h_{I_2}^{\epsilon_2}\rangle|I_1|^{-1}|I_2^{(l)}|^{-1/2}h_{I_2}^{\epsilon_2'}\sum_{J_1:J_1\subsetneq I_1}\langle b_{I_2},h_{J_1}\rangle_1 h_{J_1},\end{aligned}$$

or the symmetric one

$$\begin{aligned}&\tilde{BP}_k(a,\{b_{I_1}\}_{I_1\in\mathscr{D}^n},f)\\&:=\sum_{I_1,I_2}\langle a,h_{I_1^{(k)}}\otimes h_{I_2}\rangle\langle f,h_{I_1}^{\epsilon_1}\otimes h_{I_2}\rangle|I_1^{(k)}|^{-1/2}|I_2|^{-1}h_{I_1}^{\epsilon_1'}\sum_{J_2:J_2\subsetneq I_2}\langle b_{I_1},h_{J_2}\rangle_2 h_{J_2},\end{aligned}$$

where $a\in\mathrm{BMO}_{prod}(\mathbb{R}^n\times\mathbb{R}^m)$, and for each I_2, $b_{I_2}\in\mathrm{BMO}(\mathbb{R}^n)$ with $\sup_{I_2}\|b_{I_2}\|_{\mathrm{BMO}}<C<\infty$, similarly for $\{b_{I_1}\}$. Note that in the case when there exists some function $b^1\in\mathrm{BMO}(\mathbb{R}^n)$ such that $b_{I_2}=b^1$, $\forall I_2\in\mathscr{D}^m$, $\tilde{PB}_l$ is exactly the same as the $PB_l(a,b^1,f)$ studied in [DO16], whose boundedness has already been obtained. Similarly, $\tilde{BP}_k$ should be viewed as a generalization of the $BP_k(a,b^2,f)$ studied in [DO16]. Following exactly the same argument as in [DO16], we have the following lemma.

Lemma 4.4. *Let a, $\{b_{I_2}\}$ and $\{b_{I_1}\}$ be as described as above. Then, for any integers $k,l\geq 0$,*

$$\|\tilde{PB}_l(a,\{b_{I_2}\},f)\|_{L^2(\mathbb{R}^n\times\mathbb{R}^m)}\lesssim C\|a\|_{BMO_{prod}(\mathbb{R}^n\times\mathbb{R}^m)}\|f\|_{L^2(\mathbb{R}^n\times\mathbb{R}^m)},\quad\forall f\in L^2(\mathbb{R}^n\times\mathbb{R}^m),$$

$$\|\tilde{BP}_k(a,\{b_{I_1}\},f)\|_{L^2(\mathbb{R}^n\times\mathbb{R}^m)}\lesssim C\|a\|_{BMO_{prod}(\mathbb{R}^n\times\mathbb{R}^m)}\|f\|_{L^2(\mathbb{R}^n\times\mathbb{R}^m)},\quad\forall f\in L^2(\mathbb{R}^n\times\mathbb{R}^m).$$

When $k,l=0$, we usually omit the subscripts, simply writing $\tilde{PB}$ or $\tilde{BP}$. Now we are ready to proceed with the proof. By the definition of Π_1, the sum corresponding to the first term is equal to

$$\sum_{I_1,J_1}\sum_{I_2,J_2}\langle b,h_{I_1}\otimes h_{I_2}\rangle\langle f,h_{J_1}\otimes h_{J_2}\rangle h_{I_1}h_{J_1}^1\otimes h_{I_2}h_{J_2}^1 a_{J_1J_2}.$$

Different from the second term, here, even in the regular case, the full strength of the little BMO norm of b is needed. Specifically, in the regular case $I_1\subset J_1, I_2\subset J_2$,

$$\sum_{I_1\subset J_1,I_2\subset J_2}=\sum_{I_1=J_1,I_2=J_2}+\sum_{I_1\subsetneq J_1,I_2=J_2}+\sum_{I_1=J_1,I_2\subsetneq J_2}+\sum_{I_1\subsetneq J_1,I_2\subsetneq J_2}=:A+B+C+D.$$

Since $|a_{J_1J_2}|\leq 1$, A is a bounded paraproduct with symbol b. Moreover,

$$\begin{aligned}D&=\sum_{J_1,J_2}\langle a,h_{J_1}\otimes h_{J_2}\rangle\langle f,h_{J_1}\otimes h_{J_2}\rangle|J_1|^{-1}|J_2|^{-1}\sum_{I_1:I_1\subsetneq J_1}\sum_{I_2:I_2\subsetneq J_2}\langle b,h_{I_1}\otimes h_{I_2}\rangle h_{I_1}\otimes h_{I_2}\\&=PP(a,b,f),\end{aligned}$$

where PP is an operator studied in [DO16] whose operator norm is shown therein to be bounded by $\|a\|_{\mathrm{BMO}_{prod}}\|b\|_{\mathrm{BMO}_{prod}}$. We are thus left with B (as C is symmetric), which can be easily seen to be equal to $\tilde{PB}(a,\{b_{I_2}\},f)$ with $b_{I_2}:=\langle b,h_{I_2}\rangle_2|I_2|^{-1/2}$. Then since

$$\|\langle b,h_{I_2}\rangle_2|I_2|^{-1/2}\|_{\mathrm{BMO}}\leq\|b\|_{\mathrm{bmo}},$$

this term is bounded as well according to Lemma 4.4, which completes the discussion of the regular case.

We are left with the mixed cases of the first term, and by symmetry, it suffices to consider the case $I_1 \subset J_1, I_2 \supsetneq J_2$. Summing over I_2 gives

$$\begin{aligned}
&\sum_{I_1 \subset J_1} \sum_{I_2 \supsetneq J_2} \langle b, h_{I_1} \otimes h_{I_2} \rangle \langle f, h_{J_1} \otimes h_{J_2} \rangle h_{I_1} h^1_{J_1} \otimes h_{I_2} h^1_{J_2} a_{J_1 J_1} \\
&= \sum_{I_1 \subset J_1} \sum_{J_2} \langle b, h_{I_1} \otimes h^1_{J_2} \rangle \langle f, h_{J_1} \otimes h_{J_2} \rangle h_{I_1} h^1_{J_1} \otimes h^1_{J_2} |J_2|^{-1/2} a_{J_1 J_2} \\
&= \sum_{I_1 = J_1} + \sum_{I_1 \subsetneq J_1} =: E + F.
\end{aligned}$$

It's easily seen that

$$E = \sum_{J_1, J_2} \beta_{J_1 J_2} \langle a, h_{J_1} \otimes h_{J_2} \rangle \langle f, h_{J_1} \otimes h_{J_2} \rangle h_{J_1} \otimes h^1_{J_2} |J_1|^{-1/2} |J_2|^{-1/2}$$

with $|\beta_{J_1 J_2} := \langle b, h_{J_1} \otimes h^1_{J_2} \rangle |J_1|^{-1/2} |J_2|^{-1/2}| \le \|b\|_{\mathrm{bmo}}$, which is a bounded paraproduct with symbol a. On the other hand, $F = \tilde{PB}(a, \{b_{J_2}\}, f)$ with $b_{J_2} := \langle b, h^1_{J_2} \rangle_2 |J_2|^{-1/2}$, which is also bounded by Lemma 4.4. Therefore, the proof of Theorem 4.2 is by now complete.

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