Banach-valued $T(1)$ Type Theorems

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Joint work with Francesco Di Plinio.

Reference:
F. Di Plinio and Y. Ou, *Banach-valued Multilinear Singular Integrals*, submitted (2015).

Classical $T(1)$ theorem

Consider singular integral operator T, which maps $C_0^\infty(\mathbb{R})$ into $C_0^\infty(\mathbb{R})'$ continuously, and is associated to a standard kernel K:

$$\langle Tf, g\rangle = \int_{\mathbb{R}^2} K(x,y)f(y)g(x)\,dydx$$

for all $f, g \in C_0^\infty(\mathbb{R})$ with disjoint supports.

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- Weakly bounded:

$$|\langle T\phi_{x,t}, \psi_{x,t}\rangle| \lesssim t^{-1},$$

  for all $x \in \mathbb{R}$, $t \in (0, \infty)$, and smooth functions ϕ, ψ compactly supported on $[-1, 1]$ with sufficiently many derivatives uniformly bounded.

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- By subtracting off paraproducts, to prove the sufficiency, this theorem can be reduced to the case that $T(1) = T^*(1) = 0$.

A new core version of the theorem

Theorem (Do and Thiele, 2015)

Let $\Lambda : \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ *be a bilinear form. Let* ϕ *be some nonzero smooth function supported on* $[-1, 1]$ *with* $\int \phi = 0$*, and denote*

$$A(x, s, y, t) := \frac{\max(t, s, |y - x|)^2}{\min(t, s)}.$$

Assume that for all x, y, s, t*,*

$$A(x, s, y, t)|\Lambda(\phi_{x,s}, \phi_{y,t})| \leq C.$$

Then we have for $1 < p < \infty$*,* $|\Lambda(f, g)| \leq C_p C \|f\|_{L^p} \|g\|_{L^{p'}}$*.*

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- This implies the $T(1) = T^*(1) = 0$ version of the previous $T(1)$ theorem.

Extending Do-Thiele's result to multiparameters

Theorem (Di Plinio and Ou, 2015)

Let $\Lambda : \mathcal{S}(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ *be a bilinear form and* ϕ, ψ *be as before. Assume for all* $x_1, x_2, y_1, y_2 \in \mathbb{R}$, $s_1, s_2, t_1, t_2 > 0$ *that*

$$A(x_1, s_1, y_1, t_1)A(x_2, s_2, y_2, t_2)|\Lambda(\phi_{x_1,s_1} \otimes \psi_{x_2,s_2}, \phi_{y_1,t_1} \otimes \psi_{y_2,t_2})| \leq C.$$

Then we have for $1 < p, q < \infty$,

$$|\Lambda(f, g)| \leq C_{p,q} C \|f\|_{L^p(\mathbb{R};L^q(\mathbb{R}))} \|g\|_{L^{p'}(\mathbb{R};L^{q'}(\mathbb{R}))}.$$

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- This recovers the previously known multi-parameter $T(1)$ theorems by Journé, Martikainen, Ou, etc.
- Advantages: tensor product type testing conditions, no explicit kernel assumptions, and mixed norm estimates.

How to approach multi-parameter problems?

A useful way: treat them as vector-valued one-parameter problems and iterate. For instance, view $f(x, y) \in L^p(\mathbb{R}; L^q(\mathbb{R}))$ as $\{x \mapsto f_x(y) \in L^q(\mathbb{R})\}$, a $L^q(\mathbb{R})$-valued function in $L^p(\mathbb{R})$.

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Theorem (Di Plinio and Ou, 2015)

Let $\mathcal{X}_1, \mathcal{X}_2$ be UMD Banach spaces and $\Lambda : \mathcal{S}(\mathbb{R}; \mathcal{X}_1) \times \mathcal{S}(\mathbb{R}; \mathcal{X}_2) \to \mathbb{C}$ be a bilinear form. For any x, y, s, t define $Q_{x,s,y,t} : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathbb{C}$ as

$$Q_{x,s,y,t}(\xi, \eta) := A(x,s,y,t)\Lambda(\phi_{x,s}\xi, \phi_{y,t}\eta), \quad \forall \xi \in \mathcal{X}_1, \eta \in \mathcal{X}_2.$$

Assume that

$$\mathcal{R}_{\mathcal{X}_1,\mathcal{X}_2}(Q_{x,s,y,t} : x, y \in \mathbb{R}, s, t > 0) = C < \infty.$$

Then for $1 < p < \infty$, $|\Lambda(f,g)| \le C_p C \|f\|_{L^p(\mathbb{R};\mathcal{X}_1)} \|g\|_{L^{p'}(\mathbb{R};\mathcal{X}_2)}$.

UMD and $\mathcal{R}$-boundedness

Definition

A Banach space $\mathcal{X}$ is UMD if Hilbert transform $(H \otimes Id_{\mathcal{X}})$ is bounded on $L^p(\mathbb{R};\mathcal{X})$ for some (eq. for all) $1 < p < \infty$.

Examples: Hilbert spaces, $L^p(\nu)$, and $L^p(\mathcal{X})$ for any UMD space $\mathcal{X}$, $1 < p < \infty$.

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$\mathcal{R}$-boundedness

In general, for $Q_\lambda : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathbb{C}$, $\lambda \in \Lambda$, $\mathcal{R}_{\mathcal{X}_1,\mathcal{X}_2}(Q_\lambda : \lambda \in \Lambda)$ means:

- If $\mathcal{X}_1, \mathcal{X}_2$ are Hilbert spaces, the uniform bound of $\{Q_\lambda\}$;
- If $\mathcal{X}_1 = L^q, \mathcal{X}_2 = L^{q'}$, the least constant s.t. $\forall f_j \in L^q, g_j \in L^{q'}$,

$$\Big|\sum_{j=1}^N Q_{\lambda_j}(f_j, g_j)\Big| \leq C \Big\|\Big(\sum_{j=1}^N |f_j|^2\Big)^{1/2}\Big\|_{L^q} \Big\|\Big(\sum_{j=1}^N |g_j|^2\Big)^{1/2}\Big\|_{L^{q'}}.$$

Core idea of the proof: Banach-valued outer L^p theory

There are essentially two steps in the proof:

1. $\Lambda(f,g) \lesssim \|F(f)\|_{\text{outer } L^p} \|G(g)\|_{\text{outer } L^{p'}}$;
2. "Embedding theorems": $\|F(f)\|_{\text{outer } L^p} \lesssim \|f\|_{L^p(\mathbb{R};\mathcal{X}_1)}$.

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Outer L^p

$$\|F\|_{\text{outer } L^p} := \int_0^\infty p\lambda^{p-1} \mu(\mathrm{S}(F) > \lambda)\, d\lambda,$$

where $\mu(\mathrm{S}(F) > \lambda)$ is the smallest $\mu(E)$, such that outside E, "$F \leq \lambda$" measured by some norm S.

Banach-valued multi-parameter $T(1)$ theorem

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- By estimating paraproducts via outer embedding theorems, we also recover the full $T(1)$ theorems.
- Open: estimating multi-parameter Banach-valued paraproducts.

Thank you very much!