Commutators of Singular Integrals

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Joint work with L. Dalenc, S. Petermichl, and E. Strouse.

References:

1. Y. Ou, S. Petermichl, and E. Strouse, *Higher order Journé commutators and characterizations of multi-parameter BMO*, to appear in Adv. Math. (2015).
2. Y. Ou and S. Petermichl, *Little BMO and Journé commutators*, submitted (2015).
3. L. Dalenc and Y. Ou, *Upper bound for multi-parameter iterated commutators*, to appear in Publ. Mat. (2014).
4. Y. Ou, *Multi-parameter singular integral operators and representation theorem*, submitted (2014).

Calderón-Zgymund operators

Definition (Calderón-Zygmund (CZ) operators)

Singular integral operators that are L^2 bounded and have the kernel representation

$$Tf(x) = \int K(x,y)f(y)\,dy, \quad x \notin \operatorname{supp} f,$$

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Example (Hilbert transform)

$$Hf(x) := \lim_{\epsilon \to 0} \int_{y \in \mathbb{R}, |x-y|>\epsilon} \frac{f(y)}{x-y}\,dy.$$

Calderón-Zygmund operators

Example (Riesz transforms)

$$R_j f(x) := \lim_{\epsilon \to 0} \int_{y \in \mathbb{R}^d, |x-y| > \epsilon} f(y) \frac{x_j - y_j}{|x-y|^{d+1}} \, dy, \quad 1 \leq j \leq d$$

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Mapping properties of CZ operators:

- $L^p \to L^p$, $1 < p < \infty$;
- $L^\infty \to$ BMO; $H^1 \to L^1$.

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- $L^\infty \to \text{BMO}$; $H^1 \to L^1$.

BMO is the dual of H^1:

$$\|f\|_{\text{BMO}(\mathbb{R})} := \sup_{Q \subset \mathbb{R}} \left(\frac{1}{|Q|} \int_Q |f(x) - \langle f \rangle_Q|^2 \, dx \right)^{1/2}.$$

Outside of the scope of CZ operators

Recall that

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R_j is dilation invariant under $(x_1, \ldots, x_d) \to (\delta x_1, \ldots, \delta x_d)$, $x \in \mathbb{R}, \delta > 0$.

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Consider operators that are dilation invariant under

$$(x_1, \ldots, x_d) \to (\delta_1 x_1, \ldots, \delta_d x_d), \quad x_i \in \mathbb{R}, \delta_i > 0.$$

Example (Double Hilbert transform)

$$H \otimes H f(x) := \lim_{\epsilon_1 \to 0, \epsilon_2 \to 0} \int_{\substack{(y_1, y_2) \in \mathbb{R} \times \mathbb{R} \\ |x_1 - y_1| > \epsilon_1, |x_2 - y_2| > \epsilon_2}} \frac{f(y)}{(x_1 - y_1)(x_2 - y_2)} \, dy_1 dy_2.$$

Outside of the scope of CZ operators

Example

$$K(x,y) = \frac{x_k}{(|x|^2+y^2)^{(n+1)/2}} \cdot \frac{1}{|x|^2+iy}, \quad (x,y) \in \mathbb{R}^n \times \mathbb{R}$$

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Mapping properties:

- $T : L^p \to L^p, \forall 1 < p < \infty$;
- $T : L^\infty \to \text{BMO}_{prod}$; $H^1_{prod} \to L^1$.

Product BMO

A discrete analog of BMO:

$$\|f\|^2_{BMO^{\mathcal{D}}(\mathbb{R})} := \sup_{I_0 \subset \mathbb{R}} \frac{1}{|I_0|} \sum_{I \subset I_0, I \in \mathcal{D}} |\langle f, h_I \rangle|^2.$$

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For each dyadic interval I, $h_I := |I|^{-1/2}(\chi_{I^-} - \chi_{I^+})$, $\{h_I\}$ forms an orthonormal basis of $L^2(\mathbb{R})$.

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Definition (Dyadic product BMO)

Let $\{h_R := h_I \otimes h_J\}$ be the collection of Haar functions associated to the axes-parallel rectangles $\{R = I \times J\}$, then

$$\|f\|^2_{\mathrm{BMO}^{\mathcal{D}_1, \mathcal{D}_2}_{prod}(\mathbb{R}\times\mathbb{R})} := \sup_{\text{open } \Omega \subset \mathbb{R}\times\mathbb{R}} \frac{1}{|\Omega|} \sum_{\substack{R = I \times J \subset \Omega \\ I \in \mathcal{D}_1, J \in \mathcal{D}_2}} |\langle f, h_R \rangle|^2.$$

So what is a commutator?

Let b be a function and T be a CZ or Journé operator.

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Question

What is the relation between $\|[...[[b, T_1], T_2], ..., T_n]\|_{L^p \to L^p}$ and certain BMO norm of b? Specifically, when do we have

$$\|b\|_{\text{“BMO”}} \overset{(1)}{\lesssim} \|[...[[b, T_1], T_2], ..., T_n]\| \overset{(2)}{\lesssim} \|b\|_{\text{“BMO”}}?$$

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(1): Lower bound; (2): Upper bound. If both of them hold, $\{T_i\}$ is called a characterizing family of the corresponding “BMO”.

Known instances

Coifman-Rochberg-Weiss (1976)

$$\sup_{1\leq j\leq d} \|[b, R_j]\| \approx \|b\|_{\mathrm{BMO}};$$

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Ferguson-Sadosky (2000)

$$\|[b, H_1 \otimes H_2]\| \approx \|b\|_{\mathrm{bmo}}.$$

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Example (Weak factorization of Hardy spaces)

Let $f \in H^1(\mathbb{R})$. Then there exist $\{g_j\}, \{h_j\}$ in $L^2(\mathbb{R})$ such that $f = \sum_{j=1}^{\infty} g_j H h_j + h_j H g_j$ and $\sum_{j=1}^{\infty} \|g_j\|_2 \|h_j\|_2 \leq C\|f\|_{H^1}$.

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Example (Div-Curl estimates)

If $E, B \in L^2(\mathbb{R}^n; \mathbb{R}^n)$ such that div$E(x) =$ curl$B(x) = 0$, then

$$\|E \cdot B\|_{H^1(\mathbb{R}^n)} \leq \|E\|_{L^2(\mathbb{R}^n)} \|B\|_{L^2(\mathbb{R}^n)}.$$

Main result 1: upper bound for CZ commutators

Theorem (Dalenc-O 2014)

Let $(T_i)_{1\leq i\leq n}$ be a collection of CZ operators, with T_i acting on the i-th variable. Then,

$$\|[...[[b,T_1],T_2],...,T_n]\|_{L^p\to L^p}\leq C\|b\|_{BMO_{prod}}.$$

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Corollary

If $(T_i)_{1\leq i\leq n}$ is a characterizing family of product BMO, then its perturbation under a family of CZ operators with small enough CZ norms still characterizes product BMO.

Main result 2: new class of multi-parameter BMO spaces via "mixed" commutators

Theorem (O-Petermichl-Strouse 2015)

$$\sup_{\vec{j}} \|[[b, R_{1,j_1} \otimes R_{2,j_2}], R_{3,j_3}]\|_{L^p \to L^p} \approx \|b\|_{BMO_{(12)(3)}}.$$

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- The theorem generalizes to arbitrarily many tensor products and iterations, hence gives characterizations of a whole new family of multi-parameter BMO spaces between bmo and BMO_{prod}.
- The upper bound is a consequence of [Dalenc-O 2014]:

$$[b, T_1 T_2] = [b, T_1]T_2 + T_1[b, T_2].$$

Main result 3: upper bound for Journé commutators

Theorem (O-Petermichl-Strouse 2015)

For paraproduct free bi-parameter Journé operators T, there holds

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Theorem (O-Petermichl 2015)

Let Π be any bi-parameter standard paraproduct, then

$$\|[b, \Pi]\|_{L^p \to L^p} \lesssim \|b\|_{bmo}.$$

Representation theorems for singular integrals

Theorem (Petermichl 2000)

Hilbert transform is an average of Haar shifts

$$Sf := \sum_{I \in \mathcal{D}} \frac{1}{\sqrt{2}} \langle f, h_I \rangle h_{I^-}.$$

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Theorem (Hytönen 2010)

Any CZ operator can be represented as averages of paraproducts and dyadic shifts

$$S^{ij} f := \sum_{K \in \mathcal{D}} \sum_{\substack{I \subset K \\ \ell(I) = 2^{-i}\ell(K)}} \sum_{\substack{J \subset K \\ \ell(J) = 2^{-j}\ell(K)}} a_{IJK} \langle f, h_I \rangle h_J, \quad i, j \in \mathbb{Z}, i, j \geq 0.$$

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Generalized to bi-parameter [Martikainen 2012] and multi-parameter [O 2014].

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- $B_0 : \mathrm{BMO} \times L^p \to L^p$, $\forall 1 < p < \infty$.

Strategy of the proofs: $\|[b, T]\| \lesssim \|b\|_{\mathrm{BMO}}$

- Represent T by S^{ij} and prove $\|[b, S^{ij}]\| \lesssim (1 + \max(i, j))\|b\|_{\mathrm{BMO}}$;

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$$B_k(b, f) := \sum_I \langle b, h_{I^{(k)}}\rangle\langle f, h_I\rangle h_I |I^{(k)}|^{-1/2}, \quad k \in \mathbb{N};$$

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- Prove P and B_k, $\forall k \in \mathbb{N}$ are uniformly bounded.

Strategy of the proofs: multi-parameter cases

$\|[\ldots[[b, T_1], T_2], \ldots, T_n]\| \lesssim \|b\|_{\mathrm{BMO}_{prod}}$

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Represent T as bi-parameter dyadic shifts, and note that locally they behave like tensor products.

A few words about lower bound

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- Unlike upper bound, lower bound requires more structure on the operators; (for example, multiplication operator T never satisfies any lower bound!)
- In the Hilbert transform case, traditionally, complex analysis techniques are involved;
- In the Riesz transform case (no analytic structure any more), cone operators are introduced;
- To prove estimates of "mixed" commutators $[[b, R_{1,j_1} R_{2,j_2}], R_{3,j_3}]$, upper bound result of Journé commutators and zonal harmonics are needed.

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- Properties of the new class of multi-parameter BMO spaces?
- Weighted type estimates of commutators in connection with weighted BMO spaces?
- Complete upper bound estimate for Journé commutator? (Monster: mixed paraproducts)

Thank you very much!