Domination of multilinear singular integrals by positive sparse forms

Yumeng Ou

Brown University

Conference in harmonic analysis in honor of Michael Christ
University of Wisconsin-Madison
May 17 2016

References

Joint work with Amalia Culiuc and Francesco Di Plinio.

- A. Culiuc, F. Di Plinio and Y. Ou, *Domination of multilinear singular integrals by positive sparse forms*, submitted (2016).
- F. Di Plinio and Y. Ou, *A modulation invariant Carleson embedding theorem outside local* L^2, to appear in J. Anal. Math. (2015).

A class of multilinear multipliers

We consider multiplier forms (studied in [Muscalu-Tao-Thiele'02])

$$\Lambda_m(f_1, f_2, f_3) := \int_{\xi_1+\xi_2+\xi_3=0} m(\xi)\widehat{f_1}(\xi_1)\widehat{f_2}(\xi_2)\widehat{f_3}(\xi_3)\, d\xi,$$

with

$$\sup_{|\alpha|\leq N} \sup_{\xi_1+\xi_2+\xi_3=0} \text{dist}(\xi, \beta^\perp)^{|\alpha|} |\partial^\alpha m(\xi)| \leq C_N,$$

β: non-degenerate unit vector s.t. $\beta_1 + \beta_2 + \beta_3 = 0$.

A class of multilinear multipliers

We consider multiplier forms (studied in [Muscalu-Tao-Thiele'02])

$$\Lambda_m(f_1, f_2, f_3) := \int_{\xi_1+\xi_2+\xi_3=0} m(\xi)\widehat{f_1}(\xi_1)\widehat{f_2}(\xi_2)\widehat{f_3}(\xi_3)\, d\xi,$$

with

$$\sup_{|\alpha|\leq N} \sup_{\xi_1+\xi_2+\xi_3=0} \mathrm{dist}(\xi, \beta^{\perp})^{|\alpha|}|\partial^\alpha m(\xi)| \leq C_N,$$

β: non-degenerate unit vector s.t. $\beta_1 + \beta_2 + \beta_3 = 0$.

Example ($m(\xi) = \mathrm{sign}\,(\xi \cdot \beta)$: bilinear Hilbert transform)

A class of multilinear multipliers

We consider multiplier forms (studied in [Muscalu-Tao-Thiele'02])

$$\Lambda_m(f_1, f_2, f_3) := \int_{\xi_1+\xi_2+\xi_3=0} m(\xi)\widehat{f_1}(\xi_1)\widehat{f_2}(\xi_2)\widehat{f_3}(\xi_3)\, d\xi,$$

with

$$\sup_{|\alpha|\leq N} \sup_{\xi_1+\xi_2+\xi_3=0} \mathrm{dist}(\xi, \beta^{\perp})^{|\alpha|} |\partial^\alpha m(\xi)| \leq C_N,$$

β: non-degenerate unit vector s.t. $\beta_1 + \beta_2 + \beta_3 = 0$.

Example ($m(\xi) = \mathrm{sign}\,(\xi \cdot \beta)$: bilinear Hilbert transform)

$$BHT(f_1, f_2)(x) = p.v. \int_{\mathbb{R}} f_1(x-t) f_2(x+t)\, \frac{dt}{t}$$

A class of multilinear multipliers

We consider multiplier forms (studied in [Muscalu-Tao-Thiele'02])

$$\Lambda_m(f_1, f_2, f_3) := \int_{\xi_1+\xi_2+\xi_3=0} m(\xi)\widehat{f_1}(\xi_1)\widehat{f_2}(\xi_2)\widehat{f_3}(\xi_3)\, d\xi,$$

with

$$\sup_{|\alpha|\leq N} \sup_{\xi_1+\xi_2+\xi_3=0} \mathrm{dist}(\xi, \beta^{\perp})^{|\alpha|}|\partial^{\alpha} m(\xi)| \leq C_N,$$

β: non-degenerate unit vector s.t. $\beta_1 + \beta_2 + \beta_3 = 0$.

Example ($m(\xi) = \mathrm{sign}\,(\xi \cdot \beta)$: bilinear Hilbert transform)

$$BHT(f_1, f_2)(x) = p.v. \int_{\mathbb{R}} f_1(x-t) f_2(x+t)\, \frac{dt}{t}$$

Replacing $\mathrm{dist}(\xi, \beta^{\perp})$ by $|\xi|$ gives the Coifman-Meyer multipliers.

Domination by positive sparse forms

Theorem (Culiuc-Di Plinio-O'16)

Let $\vec{p} = (p_1, p_2, p_3)$ *s.t.* $1 < p_j < \infty, \sum_{j=1}^{3} \frac{1}{\min(p_j, 2)} < 2$. *For any* $(f_1, f_2, f_3) \in C_0^\infty(\mathbb{R})^3$ *there exists a* $\frac{1}{6}$*-sparse collection* $\mathcal{S}$ *s.t.*

$$\sup_m |\Lambda_m(f_1, f_2, f_3)| \lesssim_{\vec{p}} \mathrm{PSF}_{\mathcal{S}}^{\vec{p}}(f_1, f_2, f_3) =: \sum_{I \in \mathcal{S}} |I| \prod_{j=1}^{3} \langle f_j \rangle_{I, p_j}$$

where

$$\langle f \rangle_{I,p} := \left(\frac{1}{|I|} \int_I |f|^p \right)^{1/p}.$$

Domination by positive sparse forms

Theorem (Culiuc-Di Plinio-O'16)

Let $\vec{p} = (p_1, p_2, p_3)$ *s.t.* $1 < p_j < \infty, \sum_{j=1}^{3} \frac{1}{\min(p_j, 2)} < 2$. *For any* $(f_1, f_2, f_3) \in C_0^\infty(\mathbb{R})^3$ *there exists a* $\frac{1}{6}$*-sparse collection* $\mathcal{S}$ *s.t.*

$$\sup_m |\Lambda_m(f_1, f_2, f_3)| \lesssim_{\vec{p}} \mathrm{PSF}_{\mathcal{S}}^{\vec{p}}(f_1, f_2, f_3) =: \sum_{I \in \mathcal{S}} |I| \prod_{j=1}^{3} \langle f_j \rangle_{I, p_j}$$

where

$$\langle f \rangle_{I,p} := \left(\frac{1}{|I|} \int_I |f|^p \right)^{1/p}.$$

- $\mathcal{S}$ is a η-sparse collection of intervals if $\forall I \in \mathcal{S}, \exists E_I \subset I$ with $|E_I| \geq \eta |I|$ s.t. $\{E_I : I \in \mathcal{S}\}$ are pairwise disjoint.

Domination by positive sparse forms

Theorem (Culiuc-Di Plinio-O'16)

Let $\vec{p} = (p_1, p_2, p_3)$ *s.t.* $1 < p_j < \infty, \sum_{j=1}^{3} \frac{1}{\min(p_j, 2)} < 2$. *For any* $(f_1, f_2, f_3) \in C_0^\infty(\mathbb{R})^3$ *there exists a* $\frac{1}{6}$*-sparse collection* $\mathcal{S}$ *s.t.*

$$\sup_m |\Lambda_m(f_1, f_2, f_3)| \lesssim_{\vec{p}} \mathrm{PSF}_{\mathcal{S}}^{\vec{p}}(f_1, f_2, f_3) =: \sum_{I \in \mathcal{S}} |I| \prod_{j=1}^{3} \langle f_j \rangle_{I, p_j}$$

where

$$\langle f \rangle_{I,p} := \left(\frac{1}{|I|} \int_I |f|^p \right)^{1/p}.$$

- $\mathcal{S}$ is a η-sparse collection of intervals if $\forall I \in \mathcal{S}, \exists E_I \subset I$ with $|E_I| \geq \eta |I|$ s.t. $\{E_I : I \in \mathcal{S}\}$ are pairwise disjoint.
- Examples of $\vec{p}$: $(1^+, 2^-, 2^-)$, $(\frac{4}{3}^+, \frac{4}{3}^-, 2^-)$,...

A very brief history of positive sparse domination

Theorem (Lerner'12,..., Lacey'15, Lerner'15)

Let T be a Calderón-Zygmund (CZ) operator and $f \in C_0^\infty(\mathbb{R})$. Then there is a $\frac{1}{2}$-sparse collection $\mathcal{S}$ s.t.

$$|Tf(x)| \lesssim \mathrm{PSO}^1_{\mathcal{S}} f(x) =: \sum_{I \in \mathcal{S}} \langle f \rangle_{I,1} \chi_I(x), \quad \forall x \in \mathbb{R}.$$

A very brief history of positive sparse domination

Theorem (Lerner'12,..., Lacey'15, Lerner'15)

Let T be a Calderón-Zygmund (CZ) operator and $f \in C_0^\infty(\mathbb{R})$. Then there is a $\frac{1}{2}$-sparse collection $\mathcal{S}$ s.t.

$$|Tf(x)| \lesssim \mathrm{PSO}_{\mathcal{S}}^1 f(x) =: \sum_{I \in \mathcal{S}} \langle f \rangle_{I,1} \chi_I(x), \quad \forall x \in \mathbb{R}.$$

Theorem (Lerner-Nazarov'15, Conde-Rey'15)

Let T be a bilinear CZ operator and $f_1, f_2 \in C_0^\infty(\mathbb{R})$. Then there is a $\frac{1}{2}$-sparse collection $\mathcal{S}$ s.t.

$$|T(f_1, f_2)(x)| \lesssim \mathrm{PSO}_{\mathcal{S}}^{1,1}(f_1, f_2)(x) =: \sum_{I \in \mathcal{S}} \langle f_1 \rangle_{I,1} \langle f_2 \rangle_{I,1} \chi_I(x), \quad \forall x \in \mathbb{R}.$$

Pointwise domination in our case?

- Pointwise domination implies the domination by forms ($\mathrm{PSF}^{\vec{1}}$ is the dual form of $\mathrm{PSO}^{1,1}$).

Pointwise domination in our case?

- Pointwise domination implies the domination by forms ($\mathrm{PSF}^{\vec{1}}$ is the dual form of $\mathrm{PSO}^{1,1}$).
- Suppose $|T_m(f_1, f_2)(x)| \lesssim \sum_{I \in \mathcal{S}} \langle f_1 \rangle_{I,p_1} \langle f_2 \rangle_{I,p_2} \chi_I(x)$, then T_m will inherit certain L^1-boundedness.

Pointwise domination in our case?

- Pointwise domination implies the domination by forms ($\mathrm{PSF}^{\vec{1}}$ is the dual form of $\mathrm{PSO}^{1,1}$).
- Suppose $|T_m(f_1, f_2)(x)| \lesssim \sum_{I\in\mathcal{S}} \langle f_1\rangle_{I,p_1} \langle f_2\rangle_{I,p_2} \chi_I(x)$, then T_m will inherit certain L^1-boundedness.
- But this is false when $\inf(p_1, p_2) < 2$ and not expected otherwise. (No L^1-boundedness properties are expected to hold even for the bilinear Hilbert transform.)

Pointwise domination in our case?

- Pointwise domination implies the domination by forms ($\mathrm{PSF}^{\vec{1}}$ is the dual form of $\mathrm{PSO}^{1,1}$).
- Suppose $|T_m(f_1, f_2)(x)| \lesssim \sum_{I \in \mathcal{S}} \langle f_1 \rangle_{I,p_1} \langle f_2 \rangle_{I,p_2} \chi_I(x)$, then T_m will inherit certain L^1-boundedness.
- But this is false when $\inf(p_1, p_2) < 2$ and not expected otherwise. (No L^1-boundedness properties are expected to hold even for the bilinear Hilbert transform.)
- The difference in strength between dominating by PSO and by PSF is only formal.

Why is domination useful? L^p bounds

Corollary (Culiuc-Di Plinio-O'16; originally in Muscalu-Tao-Thiele'02)

The adjoint bilinear operators T_m to the forms Λ_m have the mapping properties

$$T_m : L^{q_1}(\mathbb{R}) \times L^{q_2}(\mathbb{R}) \to L^{\frac{q_1 q_2}{q_1+q_2}}(\mathbb{R})$$

for all (q_1, q_2) s.t. $1 < \inf(q_1, q_2) < \infty$ *and*

$$\frac{1}{q_1} + \frac{1}{q_2} < \frac{3}{2}.$$

Why is domination useful? Sharp weighted estimates

Corollary (Culiuc-Di Plinio-O'16)

Let $\vec{q} = (q_1, q_2, q_3)$ *with* $1 < q_j < \infty, \sum_{j=1}^{3} \frac{1}{q_j} = 1$ *and a weight vector* $\vec{v} = (v_1, v_2, v_3)$ *s.t.* $\prod_{j=1}^{3} v_j^{1/q_j} = 1$*. Then,*

$$\sup_m |\Lambda_m(f_1, f_2, f_3)| \le \inf_{\vec{p}} \left(C(\vec{p}, \vec{q}) [\vec{v}]_{A_{\vec{q}}^{\vec{p}}}^{\max\{\frac{q_j}{q_j - p_j}\}} \right) \prod_{j=1}^{3} \|f_j\|_{L^{q_j}(v_j)}$$

where inf is taken over open admissible tuples $\vec{p}$ *with* $p_j < q_j$ *and*

$$[\vec{v}]_{A_{\vec{q}}^{\vec{p}}} := \sup_{I \subset \mathbb{R}} \prod_{j=1}^{3} \langle v_j^{\frac{p_j}{p_j - q_j}} \rangle_I^{1/p_j - 1/q_j}.$$

An example of such weights

Corollary (Culiuc-Di Plinio-O'16)

Let $\vec{q}$ be as above and weights v_1, v_2, u be such that $u = \prod_{j=1}^{2} v_j^{q_3'/q_j}$. Assume that $v_1^2 \in A_{q_1}, v_2^2 \in A_{q_2}$. Then it holds uniformly over m that

$$T_m : L^{q_1}(v_1) \times L^{q_2}(v_2) \to L^{q_3'}(u).$$

An example of such weights

Corollary (Culiuc-Di Plinio-O'16)

Let $\vec{q}$ be as above and weights v_1, v_2, u be such that $u = \prod_{j=1}^{2} v_j^{q_3'/q_j}$. Assume that $v_1^2 \in A_{q_1}, v_2^2 \in A_{q_2}$. Then it holds uniformly over m that

$$T_m : L^{q_1}(v_1) \times L^{q_2}(v_2) \to L^{q_3'}(u).$$

In particular, $v^2 \in A_p \iff v \in A_{\frac{p+1}{2}} \cap RH_2$, where

$$[v]_{A_p} := \sup_I \left(\frac{1}{|I|} \int_I v \right) \left(\frac{1}{|I|} \int_I v^{1/(1-p)} \right)^{p-1}, \ [v]_{RH_p} := \sup_I \frac{\left(\frac{1}{|I|} \int_I v^p\right)^{1/p}}{\frac{1}{|I|} \int_I v}.$$

Why is domination useful? Vector-valued estimates

Corollary (Culiuc-Di Plinio-O'16; originally in Benea-Muscalu'15)

Let $\mathbf{m} = \{m_k\}$ *be a sequence of multipliers and*

$$T_{\mathbf{m}} : (\{f_{1k}\}, \{f_{2k}\}) \mapsto \{T_{m_k}(f_{1k}, f_{2k})\}.$$

For the tuple $\vec{r}$ *with* $1 < r_j \leq \infty, \sum_{j=1}^{3} \frac{1}{r_j} = 1$ *there holds*

$$T_{\mathbf{m}} : L^{q_1}(\mathbb{R}; \ell^{r_1}) \times L^{q_2}(\mathbb{R}; \ell^{r_2}) \to L^{\frac{q_1 q_2}{q_1 + q_2}}(\mathbb{R}; \ell^{r_3'})$$

for all (q_1, q_2) *s.t.* $1 < \inf(q_1, q_2) < \infty$ *and*

$$\sum_{j=1}^{3} \frac{1}{\min(q_j, r_j, 2)} < 2, \quad \frac{1}{q_3} := \max(1 - \frac{1}{q_1} - \frac{1}{q_2}, 0).$$

A few highlights

- Unlike previous sparse domination results, our argument doesn't rely on a priori (e.g. weak L^1) boundedness properties of the operators.

A few highlights

- Unlike previous sparse domination results, our argument doesn't rely on a priori (e.g. weak L^1) boundedness properties of the operators.
- Our domination holds uniformly for the class of multipliers (the sparse collection doesn't depend on the operator).

A few highlights

- Unlike previous sparse domination results, our argument doesn't rely on a priori (e.g. weak L^1) boundedness properties of the operators.
- Our domination holds uniformly for the class of multipliers (the sparse collection doesn't depend on the operator).
- The main ingredient of our proof is a stopping time argument that relies on intrinsic model sums involving wave packets and certain localized embedding theorems in the framework of outer measure L^p theory developed in [Do-Thiele'15].

A few highlights

- Unlike previous sparse domination results, our argument doesn't rely on a priori (e.g. weak L^1) boundedness properties of the operators.
- Our domination holds uniformly for the class of multipliers (the sparse collection doesn't depend on the operator).
- The main ingredient of our proof is a stopping time argument that relies on intrinsic model sums involving wave packets and certain localized embedding theorems in the framework of outer measure L^p theory developed in [Do-Thiele'15].
- The localized embedding theorem is derived in [Di Plinio-O'15].

A few highlights

- Unlike previous sparse domination results, our argument doesn't rely on a priori (e.g. weak L^1) boundedness properties of the operators.
- Our domination holds uniformly for the class of multipliers (the sparse collection doesn't depend on the operator).
- The main ingredient of our proof is a stopping time argument that relies on intrinsic model sums involving wave packets and certain localized embedding theorems in the framework of outer measure L^p theory developed in [Do-Thiele'15].
- The localized embedding theorem is derived in [Di Plinio-O'15].
- It seems that our approach can be applied to obtain similar sparse domination of other operators such as the variational Carleson operator and maximal truncations of bilinear singular integrals.

Thank you for your attention!