Sparse domination of singular integral operators

Yumeng Ou

MIT

MSRI Postdoc Seminar
(Colloquium style talk)
April 14 2017

Singular integral operators

Example (Hilbert transform)

$$Hf(x) = p.v.\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(y)}{x-y}\,dy, \quad \widehat{Hf}(\xi) = -i\operatorname{sgn}(\xi)\widehat{f}(\xi).$$

Singular integral operators

Example (Hilbert transform)

$$Hf(x) = p.v.\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(y)}{x-y}\,dy, \quad \widehat{Hf}(\xi) = -i\mathrm{sgn}(\xi)\widehat{f}(\xi).$$

- Convergence of Fourier series, complex analysis, signal processing...

Singular integral operators

Example (Hilbert transform)

$$Hf(x) = p.v.\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(y)}{x-y}\,dy, \quad \widehat{Hf}(\xi) = -i\text{sgn}(\xi)\widehat{f}(\xi).$$

- Convergence of Fourier series, complex analysis, signal processing...
- The prototype operator in the theory of singular integrals:
  - Higher dimensional: Riesz transforms $\widehat{R_j f}(\xi) = \frac{\xi_j}{|\xi|}\widehat{f}(\xi)$.
  - Fourier multipliers: $\widehat{Tf}(\xi) = m(\xi)\widehat{f}(\xi)$.
  - (Non-convolutional) Calderón-Zygmund operators:

$$Tf(x) = \int K(x,y)f(y)\,dy.$$

  - More: Radon transforms, oscillatory integrals, multiparameter operators, multilinear singular integrals, discrete analogs, etc.

Fundamental question: mapping properties of operators

- $T: L^p(\mathbb{R}) \to L^p(\mathbb{R})$?
- $T: L^p(\mathbb{R}) \to L^{p,\infty}(\mathbb{R})$? $(\|f\|_{L^{p,\infty}} := \sup_{\lambda>0} \lambda \left|\{x : |f(x)| > \lambda\}\right|^{1/p})$

Fundamental question: mapping properties of operators

- $T: L^p(\mathbb{R}) \to L^p(\mathbb{R})$?
- $T: L^p(\mathbb{R}) \to L^{p,\infty}(\mathbb{R})$? $(\|f\|_{L^{p,\infty}} := \sup_{\lambda>0} \lambda |\{x : |f(x)| > \lambda\}|^{1/p})$
- $T: L^p(w) \to L^p(w)$ for positive $w \in L^1_{\text{loc}}$? $(\|f\|_{L^p(w)} := \int |f|^p w)$

Example (Hunt-Muckenhoupt-Wheeden 1973)

$\forall 1 < p < \infty$, $H : L^p(w) \to L^p(w)$ iff $w \in A_p$, i.e.

$$[w]_{A_p} := \sup_Q \langle w \rangle_Q \langle w^{1-p'} \rangle_Q^{p-1} := \sup_Q \left(\fint_Q w \right) \left(\fint_Q w^{1-p'} \right)^{p-1} < \infty.$$

- If $T: L^p(w) \to L^p(w)$, what is the sharp behavior of the operator norm $C([w]_{A_p})$?

Calderón-Zygmund operators

CZ operator (including the Hilbert transform)

L^2 bounded, standard kernel:

$$Tf(x) = \int K(x,y)f(y)\,dy \quad x \notin \operatorname{spt} f.$$

Calderón-Zygmund operators

CZ operator (including the Hilbert transform)

L^2 bounded, standard kernel:

$$Tf(x) = \int K(x,y) f(y)\, dy \quad x \notin \operatorname{spt} f.$$

- $T: L^p \to L^p$ $(1 < p < \infty)$, $L^1 \to L^{1,\infty}$.
- $T: L^p(w) \to L^p(w)$ for all $w \in A_p$, $\forall 1 < p < \infty$.

Theorem (Petermichl 2004 (Hilbert transform); Hytönen 2010 (general CZ operators))

$$\|Tf\|_{L^p(w)} \leq C_p [w]_{A_p}^{\max(1, 1/(p-1))} \|f\|_{L^p(w)}, \quad \forall 1 < p < \infty.$$

Sparse: a finer quantification of boundedness

Theorem (*)

CZ operator T satisfies "$Tf \lesssim \mathcal{A}_{\mathcal{S}} f$", where

$$\mathcal{A}_{\mathcal{S}}(f) := \sum_{Q \in \mathcal{S}} \langle |f| \rangle_Q \chi_Q$$

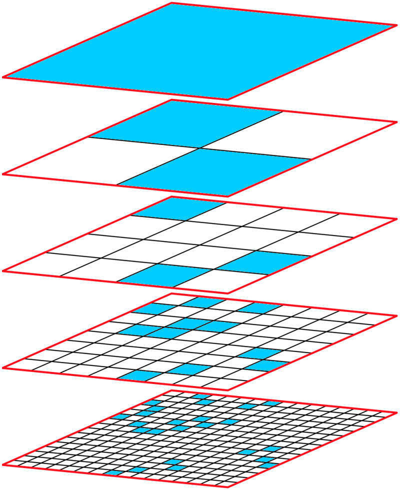
for some sparse collection $\mathcal{S}$ of cubes.

$\mathcal{S}$ is sparse if $\forall Q \in \mathcal{S}$, $\exists E_Q \subset Q$ such that $|E_Q| > \frac{1}{2}|Q|$ and $\{E_Q : Q \in \mathcal{S}\}$ pairwise disjoint.

Corollary

CZ operator T maps $L^p \to L^p \, (1 < p < \infty)$, $L^1 \to L^{1,\infty}$, and satisfies the sharp A_p bound.

A sparse collection of cubes ([Lerner-Nazarov 2015])



Interpret "$Tf \lesssim \mathcal{A}_{\mathcal{S}} f$"

- Given f, there exists $\mathcal{S} = \mathcal{S}(f, T)$ such that
  - Lerner 2013: $\|Tf\|_X \lesssim \|\mathcal{A}_{\mathcal{S}} f\|_X$;
  - Conde-Rey, Lerner-Nazarov, Lacey, Hytönen-Roncal-Tapiola, Lerner 2015: $Tf(x) \lesssim \sum_{j=1}^{3^n} \mathcal{A}_{\mathcal{S}_j} f(x)$ pointwisely;
- Culiuc-Di Plinio-O. 2016: Given f, g, there exists $\mathcal{S} = \mathcal{S}(f, g)$ such that for all CZ operator T,

$$|\langle Tf, g\rangle| \leq C_T \Lambda_{\mathcal{S},1,1}(f, g) := \langle \mathcal{A}_{\mathcal{S}} f, g\rangle = \sum_{Q\in\mathcal{S}} |Q| \langle f\rangle_Q \langle g\rangle_Q;$$

- Lacey-Mena 2016: same result as above under weaker ($T1$ type) assumptions on the operators.

To recap

$$\underbrace{|\langle Tf, g\rangle|}_{\text{messy, complicated}} \leq C \sup_{\Lambda} \underbrace{\Lambda_{\mathcal{S},1,1}(f,g)}_{\text{positive, localized}}$$

To recap

$$\underbrace{|\langle Tf, g\rangle|}_{\text{messy, complicated}} \leq C \sup_{\Lambda} \underbrace{\Lambda_{\mathcal{S},1,1}(f,g)}_{\text{positive, localized}}$$

How sparse bound implies L^p bound?

$$\Lambda_{\mathcal{S},1,1}(f,g) = \sum_{Q\in\mathcal{S}} |Q|\langle f\rangle_Q \langle g\rangle_Q \leq C \sum_{Q\in\mathcal{S}} |E_Q|\langle f\rangle_Q\langle g\rangle_Q$$

$$\leq C \sum_{Q\in\mathcal{S}} |E_Q| \inf_{x\in E_Q} Mf(x)Mg(x) \leq C \sum_{Q\in\mathcal{S}} \int_{E_Q} (Mf)(Mg)$$

$$\leq C \int_{\mathbb{R}^n} (Mf)(Mg) \leq C\|Mf\|_{L^p}\|Mg\|_{L^{p'}} \leq C'\|f\|_{L^p}\|g\|_{L^{p'}}.$$

Toy sparse bound: identity operator

Theorem

Given f supported in Q_0, $\exists$ sparse collection $\mathcal{S}$ of dyadic cubes such that

$$f(x) \lesssim \sum_{Q \in \mathcal{S}} \langle f \rangle_Q \chi_Q(x), \quad \text{a.e. } x \in Q_0.$$

Toy sparse bound: identity operator

Theorem

Given f supported in Q_0, $\exists$ sparse collection $\mathcal{S}$ of dyadic cubes such that

$$f(x) \lesssim \sum_{Q\in\mathcal{S}} \langle f\rangle_Q \chi_Q(x), \quad \text{a.e. } x \in Q_0.$$

This follows from iterating the Calderón-Zygmund decomposition:

- Let $\mathcal{E}$ be the collection of maximal dyadic cubes $Q \subset Q_0$ s.t.

$$\langle f\rangle_Q > C\langle f\rangle_{Q_0}.$$

- Split $f = \underbrace{f\chi_{Q_0\setminus(\bigcup_{Q\in\mathcal{E}} Q)}}_{\leq C\langle f\rangle_{Q_0}} + \sum_{Q\in\mathcal{E}} \underbrace{f\chi_Q}_{\text{recurse}}$.
- Choose C sufficiently large s.t. $\sum_{Q\in\mathcal{E}} |Q| < \epsilon|Q_0| \implies$ sparsity.

Toy sparse bound: Haar multiplier

Theorem

Given f_1, f_2, $\exists$ sparse collection $\mathcal{S}$ of dyadic cubes such that

$$\langle Tf_1, f_2\rangle := \left\langle \sum_{I\in\mathcal{D}} \epsilon_I \langle f_1, h_I\rangle h_I, f_2 \right\rangle \lesssim \sum_{Q\in\mathcal{S}} |Q| \langle f_1\rangle_Q \langle f_2\rangle_Q .$$

Toy sparse bound: Haar multiplier

Theorem

Given f_1, f_2, $\exists$ sparse collection $\mathcal{S}$ of dyadic cubes such that

$$\langle Tf_1, f_2\rangle := \left\langle \sum_{I\in\mathcal{D}} \epsilon_I \langle f_1, h_I\rangle h_I, f_2 \right\rangle \lesssim \sum_{Q\in\mathcal{S}} |Q| \langle f_1\rangle_Q \langle f_2\rangle_Q.$$

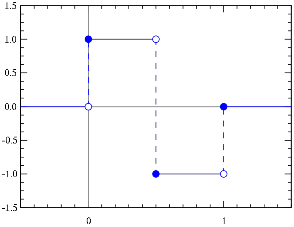


Figure: $h_{[0,1)}$ (Wikipedia)

- Haar function $h_I := |I|^{-1/2}(\chi_{I^-} - \chi_{I^+})$.
- $\{h_I\}_I$ forms an orthonormal basis of L^2:

$$f = \sum_{I\in\mathcal{D}} \langle f, h_I\rangle h_I.$$

- $\langle Tf_1, f_2\rangle = \sum_{I\subset Q_0} \epsilon_I \langle f_1, h_I\rangle \langle f_2, h_I\rangle$.

Iterate the following argument

Find collection $\mathcal{E}$ of pairwise disjoint cubes s.t. $\sum_{Q\in\mathcal{E}} |Q| < \epsilon|Q_0|$ and

$$\sum_{I\subset Q_0} \epsilon_I \langle f_1, h_I\rangle\langle f_2, h_I\rangle = \underbrace{\Lambda(f_1, f_2)}_{\leq C|Q_0|\langle f_1\rangle_{Q_0}\langle f_2\rangle_{Q_0}} + \sum_{Q\in\mathcal{E}} \underbrace{\sum_{I\subset Q} \epsilon_I \langle f_1, h_I\rangle\langle f_2, h_I\rangle}_{\text{recurse}}.$$

Iterate the following argument

Find collection $\mathcal{E}$ of pairwise disjoint cubes s.t. $\sum_{Q\in\mathcal{E}}|Q|<\epsilon|Q_0|$ and

$$\sum_{I\subset Q_0}\epsilon_I\langle f_1,h_I\rangle\langle f_2,h_I\rangle=\underbrace{\Lambda(f_1,f_2)}_{\leq C|Q_0|\langle f_1\rangle_{Q_0}\langle f_2\rangle_{Q_0}}+\sum_{Q\in\mathcal{E}}\underbrace{\sum_{I\subset Q}\epsilon_I\langle f_1,h_I\rangle\langle f_2,h_I\rangle}_{\text{recurse}}.$$

- CZ decomposition: we can find such $\mathcal{E}$ and split for $i=1,2$

$$f_i=g_i+\sum_{Q\in\mathcal{E}}b_{i,Q},\quad \|g_i\|_{L^\infty}\lesssim\langle f_i\rangle_{Q_0},\ \operatorname{supp}b_{i,Q}\subset Q,\ \int b_{i,Q}=0.$$

- $\Lambda(g_1,b_2)=\Lambda(b_1,g_2)=\Lambda(b_1,b_2)=0$, and

$$\Lambda(g_1,g_2)\leq\|g_1\|_{L^2}\|g_2\|_{L^2}\leq\|g_1\|_{L^\infty}\|g_2\|_{L^\infty}|Q_0|\leq C|Q_0|\langle f_1\rangle_{Q_0}\langle f_2\rangle_{Q_0}.$$

Toy sparse bound: Haar shift and beyond

Theorem

Given f_1, f_2, $\exists$ sparse collection $\mathcal{S}$ of dyadic cubes such that

$$\left\langle \sum_{I\in\mathcal{D}} \epsilon_I \langle f_1, h_I\rangle h_{I^-}, f_2 \right\rangle \lesssim \sum_{Q\in\mathcal{S}} |Q| \langle f_1\rangle_Q \langle f_2\rangle_Q .$$

Toy sparse bound: Haar shift and beyond

Theorem

Given f_1, f_2, $\exists$ sparse collection $\mathcal{S}$ of dyadic cubes such that

$$\left\langle \sum_{I \in \mathcal{D}} \epsilon_I \langle f_1, h_I \rangle h_{I^-}, f_2 \right\rangle \lesssim \sum_{Q \in \mathcal{S}} |Q| \langle f_1 \rangle_Q \langle f_2 \rangle_Q.$$

- This (together with its higher complexity version) implies the sparse bound for Hilbert transform (and general CZ operators).

Toy sparse bound: Haar shift and beyond

Theorem

Given f_1, f_2, $\exists$ sparse collection $\mathcal{S}$ of dyadic cubes such that

$$\left\langle \sum_{I\in\mathcal{D}} \epsilon_I \langle f_1, h_I\rangle h_{I^-}, f_2 \right\rangle \lesssim \sum_{Q\in\mathcal{S}} |Q| \langle f_1\rangle_Q \langle f_2\rangle_Q.$$

- This (together with its higher complexity version) implies the sparse bound for Hilbert transform (and general CZ operators).
- Main ingredients: iteration algorithm via CZ decomposition, the study of $\Lambda(g, b), \Lambda(b, b)$ using properties of the operators.

Sparse form with bumps

Question: can we study operators that are not bounded on L^p for all $1 < p < \infty$?

Sparse form with bumps

Question: can we study operators that are not bounded on L^p for all $1 < p < \infty$?

Definition

Given $1 \leq r, s < \infty$ and a sparse collection $\mathcal{S}$, the associated (r, s)-sparse form is

$$\Lambda_{\mathcal{S},r,s}(f,g) = \sum_{Q\in\mathcal{S}} |Q| \langle f \rangle_{Q,r} \langle g \rangle_{Q,s} := \sum_{Q\in\mathcal{S}} |Q| \left(\fint_Q |f|^r \right)^{1/r} \left(\fint_Q |g|^s \right)^{1/s}.$$

For sublinear operator T, $\|T : (r,s)\|$ denotes the smallest constant C so that for all $f, g \in C_0^\infty$,

$$|\langle Tf, g\rangle| \leq C \sup_{\mathcal{S}} \Lambda_{\mathcal{S},r,s}(f,g).$$

$\|T:(r,s)\| < \infty$ implies

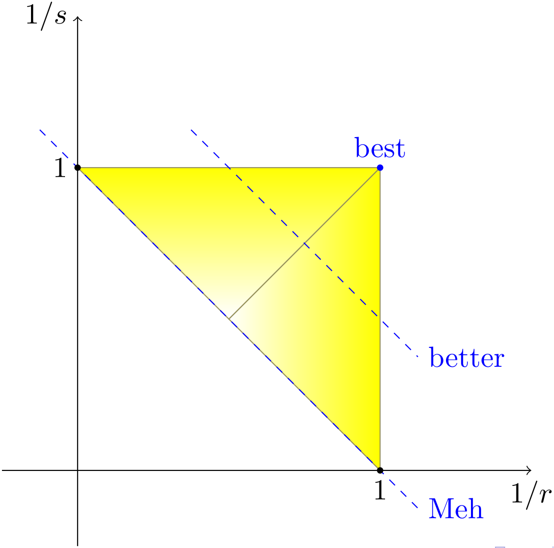
- $T: L^p \to L^p, \forall p \in (r, s'), L^r \to L^{r,\infty}$.
- $T: L^p(w) \to L^p(w)$ for $r < p < s'$ and $w \in A_{p/r} \cap \mathrm{RH}_{(s'/p)'}$, where

$$[w]_{\mathrm{RH}_q} := \text{smallest const.}\, C\, \text{s.t.} \left(\fint_Q w^q \right)^{1/q} \leq C \left(\fint_Q w \right).$$

To summarize

$(1,1)$ sparse bound is everything, (r, r') sparse bound is nothing.

“Sparse” region (picture by M. Lacey)



Beyond CZ: rough singular integrals

Example (Rough homogeneous singular integrals)

Let $\Omega \in L^1(S^{d-1})$ such that $\int_{S^{d-1}} \Omega = 0$,

$$T_\Omega f(x) = p.v. \int_{\mathbb{R}^d} f(x-y) \frac{\Omega(y/|y|)}{|y|^d}\, dy.$$

Example (Bochner-Riesz multipliers)

$$BR_\delta f = \mathcal{F}^{-1}\left[\mathcal{F}(f)(\xi)(1-|\xi|^2)_+^\delta\right].$$

Beyond CZ: rough singular integrals

Theorem (Conde-Culiuc-Di Plinio-O. 2016)

Let T be either the rough homogeneous singular integral T_Ω with $\Omega \in L^\infty(S^{d-1})$ or the Bochner-Riesz multiplier BR_δ at the critical index $\delta = \frac{d-1}{2}$, then

$$\|T : (1,p)\| \le \frac{Cp}{p-1}, \quad \forall 1 < p < \infty.$$

Beyond CZ: rough singular integrals

Theorem (Conde-Culiuc-Di Plinio-O. 2016)

Let T be either the rough homogeneous singular integral T_Ω with $\Omega \in L^\infty(S^{d-1})$ or the Bochner-Riesz multiplier BR_δ at the critical index $\delta = \frac{d-1}{2}$, then

$$\|T : (1,p)\| \leq \frac{Cp}{p-1}, \quad \forall 1 < p < \infty.$$

For $\Omega \in L^{q,1}\log L(S^{d-1})$, the Orlicz-Lorentz space, $1 < q < \infty$,

$$\|T_\Omega : (1,p)\| \leq \frac{Cp}{p-1}, \quad \forall p \geq q'.$$

Remark: $L^{q+\epsilon} \hookrightarrow L^{q,1}\log L \hookrightarrow L^q$.

An abstract theorem (Conde-Culiuc-Di Plinio-O. 2016)

Given $B(f,g) = \int_{\mathbb{R}^d \times \mathbb{R}^d} K(x,y) f(x) g(y)\, dxdy$, if $\operatorname{spt} f \cap \operatorname{spt} g = \emptyset$.
Suppose $K = \sum_{s \in \mathbb{Z}} K_s$ with

$$\operatorname{spt} K_s \subset \left\{ (x,y) \in \mathbb{R}^d \times \mathbb{R}^d : 2^{s-2} < |x-y| < 2^s \right\}.$$

Then $\|B : (\max\{r, q'\}, \max\{s, q'\})\| < \infty$ if the following hold:

1. $\sup_{x \in \mathbb{R}^d} \left(\|K_s(x, x+\cdot)\|_q + \|K_s(x+\cdot, x)\|_q \right) < 2^{\frac{-sd}{q'}}$, $1 < q \leq \infty$.
2. The truncated forms B_μ^ν (associated to $\sum_{\mu < s \leq \nu} K_s$) are uniformly bounded on $L^p \times L^{p'}$, for some $1 < p < \infty$.
3. For any stopping collection $\mathcal{Q} = \{L : L \subset Q\}$ of cubes,

$$\left| B^{\log(\ell(Q))}(b_1 \chi_Q, g_2) - \sum_{L \in \mathcal{Q}} B^{\log(\ell(L))}(b_1 \chi_L, g_2) \right| \lesssim |Q| \langle f \rangle_{Q,r} \langle g \rangle_{Q,s}.$$

Remark on the last condition:

Earlier sparse proofs usually use a stronger version: the weak $(1,1)$ bound of the grand maximal truncated operator

$$\mathcal{M}_T f(x) := \sup_{Q \ni x} \operatorname*{esssup}_{\xi \in Q} |T(f\chi_{(3Q)^c})(\xi)|$$

Remark on the last condition:

Earlier sparse proofs usually use a stronger version: the weak $(1,1)$ bound of the grand maximal truncated operator

$$\mathcal{M}_T f(x) := \sup_{Q \ni x} \operatorname*{esssup}_{\xi \in Q} |T(f\chi_{(3Q)^c})(\xi)|$$

- We only use CZ decomposition, hence obtain a sparse collection $\mathcal{S}$ independent of T.
- Sometimes it is difficult or impossible to have the weak $(1,1)$ bound of $\mathcal{M}_T$: it's still open whether the maximal truncation of T_Ω is weak $(1,1)$ even for $\Omega \in L^\infty$.

Corollaries and comparison to other results

- T_Ω and $BR_{\frac{d-1}{2}}$ map L^1 into $L^{1,\infty}$ (Seeger 1996; Christ 1988).
- If $\Omega \in L^{q,1}\log L(S^{d-1})$ for some $1 < q < \infty$, then

$$\|T_\Omega : L^p(w) \to L^p(w)\| \lesssim [w]_{A_{p/q'}}^{\max(1,(p-q')^{-1})}, \quad \forall q' < p < \infty$$

- If $T = T_\Omega$ for $\Omega \in L^\infty(S^{d-1})$ or $T = BR_{\frac{d-1}{2}}$, then

$$\|T : L^p(w) \to L^p(w)\| \lesssim [w]_{A_p}^{\frac{1}{p-1}\max(p,2)}, \quad \forall 1 < p < \infty.$$

Corollaries and comparison to other results

- T_Ω and $BR_{\frac{d-1}{2}}$ map L^1 into $L^{1,\infty}$ (Seeger 1996; Christ 1988).
- If $\Omega \in L^{q,1}\log L(S^{d-1})$ for some $1 < q < \infty$, then

$$\|T_\Omega : L^p(w) \to L^p(w)\| \lesssim [w]_{A_{p/q'}}^{\max(1,(p-q')^{-1})}, \quad \forall q' < p < \infty$$

- If $T = T_\Omega$ for $\Omega \in L^\infty(S^{d-1})$ or $T = BR_{\frac{d-1}{2}}$, then

$$\|T : L^p(w) \to L^p(w)\| \lesssim [w]_{A_p}^{\frac{1}{p-1}\max(p,2)}, \quad \forall 1 < p < \infty.$$

- Hytönen-Roncal-Tapiola 2015; Benea-Bernicot-Luque 2016; Li-Pérez-Rivera-Roncal 2017.

Beyond CZ: bilinear Hilbert transform

$$
\begin{aligned}
BHT(f,g) :=& p.v. \int_{\mathbb{R}} f(x-t)g(x+t)\, \frac{dt}{t} \\
=& C \int_{\mathbb{R}^2} \operatorname{sgn}(\xi-\eta) \widehat{f}(\xi)\widehat{g}(\eta) e^{2\pi i x(\xi+\eta)}\, d\xi d\eta.
\end{aligned}
$$

Beyond CZ: bilinear Hilbert transform

$$\begin{aligned} BHT(f,g) :=& p.v. \int_{\mathbb{R}} f(x-t)g(x+t)\,\frac{dt}{t} \\ =& C\int_{\mathbb{R}^2} \operatorname{sgn}(\xi-\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x(\xi+\eta)}\,d\xi d\eta. \end{aligned}$$

Theorem (Culiuc-Di Plinio-O. 2016)

For admissible (p_1, p_2, p_3), i.e. $1 < p_j < \infty$, $\sum_{j=1}^{3} \frac{1}{\min(p_j,2)} < 2$ (for instance $(2^-, 2^-, 1^+)$),

$$\|\langle BHT(f_1, f_2), f_3\rangle : (p_1, p_2, p_3)\| < \infty.$$

Trilinear sparse form $\Lambda_{\mathcal{S},\vec{p}}(f_1, f_2, f_3) := \sum_{Q\in\mathcal{S}} |Q| \prod_{j=1}^{3} \langle f_j\rangle_{Q,p_j}$.

Beyond CZ: bilinear Hilbert transform

Corollary (Sample)

Let $1 < q_j < \infty$ be so that $\sum_{j=1}^{3} \frac{1}{q_j} = 1$ and weights v_1, v_2, u satisfy $u = v_1^{q_3'/q_1} v_2^{q_3'/q_2}$. Assume that $v_1^2 \in A_{q_1}, v_2^2 \in A_{q_2}$. Then

$$BHT : L^{q_1}(v_1) \times L^{q_2}(v_2) \to L^{q_3'}(u).$$

In particular, $v^2 \in A_p \iff v \in A_{\frac{p+1}{2}} \cap RH_2$.

Beyond CZ: bilinear Hilbert transform

Corollary (Sample)

Let $1 < q_j < \infty$ be so that $\sum_{j=1}^{3} \frac{1}{q_j} = 1$ and weights v_1, v_2, u satisfy $u = v_1^{q_3'/q_1} v_2^{q_3'/q_2}$. Assume that $v_1^2 \in A_{q_1}, v_2^2 \in A_{q_2}$. Then

$$BHT : L^{q_1}(v_1) \times L^{q_2}(v_2) \to L^{q_3'}(u).$$

In particular, $v^2 \in A_p \iff v \in A_{\frac{p+1}{2}} \cap RH_2$.

A general framework using Do-Thiele's outer measure theory

$$\underbrace{\text{Outer Hölder's inequality}}_{\text{multilinear} \to \text{linear}} + \underbrace{\text{Embedding theorems}}_{\text{outer } L^p \to L^p} \implies \text{Sparse bound}$$

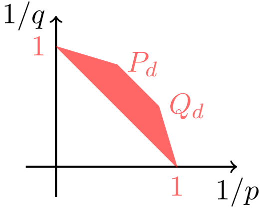
Beyond CZ: Hilbert transform along curves

$$H_\gamma f(x) := p.v. \int_{\mathbb{R}} f(x-\gamma(t))\,\frac{dt}{t}, \quad \gamma = (t, t^2, \dots, t^d).$$

Theorem (2017)

For $(1/p, 1/q)$ in the L^p improving region below, where $P_d = (\frac{2}{d+1}, \frac{d^2-d+2}{d^2+d}), Q_d = (\frac{d^2-d+2}{d^2+d}, \frac{2}{d+1})$,

$$\|H_\gamma : (p,q)\| < \infty.$$



Beyond CZ: some other sparse breakthroughs

- Lacey 2017: spherical maximal functions
- Barron 2017: rough bilinear singular integrals
- Bernicot-Frey-Petermichl 2016: non-integral singular operators
- Lacey-Spencer 2017, Krause-Lacey 2017: (maximally truncated) oscillatory integrals
- Culiuc-Kesler-Lacey 2016, Kesler-Mena 2017: discrete Hilbert transforms

Beyond CZ: some other sparse breakthroughs

- Lacey 2017: spherical maximal functions
- Barron 2017: rough bilinear singular integrals
- Bernicot-Frey-Petermichl 2016: non-integral singular operators
- Lacey-Spencer 2017, Krause-Lacey 2017: (maximally truncated) oscillatory integrals
- Culiuc-Kesler-Lacey 2016, Kesler-Mena 2017: discrete Hilbert transforms

Thank you for your attention!