A cone restriction estimate using polynomial partitioning

Yumeng Ou

MIT

Recent developments in harmonic analysis
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Fourier restriction (extension) estimate

(Restriction) problem

Let S be a smooth compact hypersurface with surface measure $d\sigma$, find the optimal range of (p, q) s.t.

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- $f \in L^1 \implies \hat{f}$ is continuous, bounded $\implies$ all $q \in [1, \infty]$ work.
- $f \in L^2 \implies \hat{f} \in L^2 \implies$ all $q \in [1, \infty]$ fail.

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Equivalent (extension) problem

$$\|E_S g\|_{L^{p'}(\mathbb{R}^n)} \lesssim \|g\|_{L^{q'}(S;d\sigma)}, \quad E_S g(x) := \int_S g(\xi) e^{ix\cdot\xi}\, d\sigma(\xi).$$

Fourier restriction (extension) estimate

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- S needs to contain sufficient curvature. Prototype examples: compact subsets of sphere, paraboloid, cone:

$$\mathcal{C} := \{(\xi, \xi_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \xi_n = |\xi|,\ 1 \leqslant \xi_j \leqslant 2, \forall j\},$$

$$Ef(x) := E_{\mathcal{C}} f = \int_{2B^{n-1} \setminus B^{n-1}} e^{i(x_1\xi_1 + \cdots x_{n-1}\xi_{n-1} + x_n|\xi|)} f(\xi)\, d\xi.$$

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Cone restriction conjecture

For all $p > \frac{2(n-1)}{n-2}, q' \leqslant \frac{n-2}{n} p$, there holds

$$\|Ef\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^q(2B^{n-1} \setminus B^{n-1})}.$$

Conjecture: $\|Ef\|_p \lesssim \|f\|_q$ for $p > \frac{2(n-1)}{n-2}, q' \leqslant \frac{n-2}{n}p$

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Theorem (O-Wang '17)

For $n = 5$, the cone restriction estimate holds in the full conjectured range $p > \frac{8}{3}$, $q' \leqslant \frac{3}{5} p$.

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Theorem (O-Wang '17)

For $n \geqslant 3$, there holds $\|Ef\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(2B^{n-1}\setminus B^{n-1})}$ whenever

$$p > \begin{cases} 4 & \text{if } n = 3, \\ 2 \cdot \frac{3n+1}{3n-3} & \text{if } n > 3 \text{ odd}, \\ 2 \cdot \frac{3n}{3n-4} & \text{if } n > 3 \text{ even}. \end{cases}$$

Reduce to "k-broad" restriction estimate

By "ϵ-removal", it suffices to prove the localized version

$$\|Ef\|_{L^p(B_R)} \lesssim R^{\epsilon}\|f\|_{L^q(2B^{n-1}\setminus B^{n-1})}.$$

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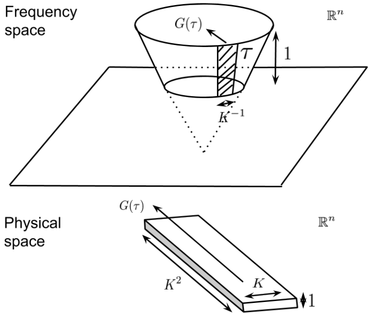
Decomposing cone $\mathcal{C}$ into strips τ with radius K^{-1} yields

$$Ef = \sum_\tau Ef_\tau, \quad \#\{\tau\} \lesssim K^{n-2} \quad (K < R^\epsilon).$$



- Frequency: τ is contained in a rectangular box of sidelengths $1 \times K^{-1} \times \cdots \times K^{-1} \times K^{-2}$;
- Space: Ef_τ is essentially supported and constant in its dual tubes of sidelengths $1 \times K \times \cdots \times K \times K^2$. $G(\tau)$: long direction of the tubes.

Reduce to "k-broad" restriction estimate

Decompose B_R into small balls B_{K^2}. For each B_{K^2}, let $V \subset \mathbb{R}^n$: $(k-1)$ dim subspace $(2 \leqslant k \leqslant n)$.

$$\|Ef\|_{L^p(B_{K^2})} \lesssim \underbrace{\Big\| \sum_{\text{“}\tau \in V\text{”}} Ef_\tau \Big\|_{L^p(B_{K^2})}}_{\text{Narrow}} + \underbrace{\Big\| \sum_{\text{“}\tau \notin V\text{”}} Ef_\tau \Big\|_{L^p(B_{K^2})}}_{\text{Broad}}.$$

"$\tau \in V$": $G(\tau)$ is close to V up to angle K^{-1}.

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1). Narrow part: decoupling + Hölder's inequality

$$\|\sum_{\text{“}\tau\in V\text{”}} Ef_\tau\|^p_{L^p(B_{K^2})} \lesssim K^\delta \left(\sum_{\text{“}\tau\in V\text{”}} \|Ef_\tau\|^2_{L^p(B_{K^2})}\right)^{p/2} \leqslant K^{C(\delta,k,p)} \sum_{\text{“}\tau\in V\text{”}} \|Ef_\tau\|^p_{L^p(B_{K^2})}.$$

Reduce to "k-broad" restriction estimate

Sum over B_{K^2}:

$$\| \sum_{\text{“}\tau\in V\text{”}} Ef_\tau\|^p_{L^p(B_R)} \lesssim K^{C(\delta,k,p)} \sum_{\tau} \|Ef_\tau\|^p_{L^p(B_R)}.$$

1. Lorentz rescaling: blow up the scale of τ, shrink the scale of B_R.
2. Induct on R: induction closes when $p > p_1(k,n)$.

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2) Broad part: "k-broad" restriction estimate for $p > p_2(k,n)$

$$\sum_{B_{K^2} \subset B_R} \| \sum_{\text{“}\tau \notin V\text{”}} Ef_\tau \|_{L^p(B_{K^2})}^p \lesssim \sum_{B_{K^2} \subset B_R} K^{O(1)} \max_{\text{“}\tau \notin V\text{”}} \|Ef_\tau\|_{L^p(B_{K^2})}^p$$
$$\lesssim C(K,\epsilon) R^{p\epsilon} \|f\|_{L^q}^p.$$

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$$\lesssim C(K,\epsilon) R^{p\epsilon} \|f\|^p_{L^q}.$$

3) Find the best k that balances $p_1(k,n), p_2(k,n)$.

k-broad norm

$$\mu_{Ef}(B_{K^2}) := \min_{V_1,\ldots,V_A (k-1)\text{-subspace of } \mathbb{R}^n} \left(\max_{\tau \notin V_a, \forall a} \|Ef_\tau\|^p_{L^p(B_{K^2})} \right)$$

gives rise to a measure on any open set U

$$\|Ef\|^p_{BL^p_{k,A}(U)} := \mu_{Ef}(U) := \sum_{B_{K^2} \subset U} \mu_{Ef}(B_{K^2}).$$

Theorem (O-Wang '17)

For any $2 \leqslant k \leqslant n$, $\epsilon > 0$, there is a large constant A s.t.

$$\|Ef\|_{BL^p_{k,A}(B_R)} \lesssim_{K,\epsilon} R^\epsilon \|f\|_{L^2(2B^{n-1} \setminus B^{n-1})}$$

holds for all K and $p \geqslant \bar{p}(k,n) = 2 \cdot \frac{n+k}{n+k-2}$.

Why need so many $(k-1)$-subspace $V_1, \ldots, V_A$?

Unlike L^p, $\|Ef\|_{BL^p_{k,A}(U)}$ is not literally a norm. But for A sufficiently large, it satisfies for all $A = A_1 + A_2$

- Triangle inequality

$$\|E(f_1 + f_2)\|_{BL^p_{k,A}(U)} \lesssim \|Ef_1\|_{BL^p_{k,A_1}(U)} + \|Ef_2\|_{BL^p_{k,A_2}(U)}.$$

- Hölder's inequality

$$\|Ef\|_{BL^p_{k,A}(U)} \leqslant \|Ef\|^{\alpha}_{BL^{p_1}_{k,A_1}(U)} \|Ef\|^{1-\alpha}_{BL^{p_2}_{k,A_2}(U)},$$

where $1 \leqslant p, p_1, p_2 < \infty$, $0 \leqslant \alpha \leqslant 1$, $\frac{1}{p} = \frac{\alpha}{p_1} + \frac{1-\alpha}{p_2}$.

k-broad V.S. k-linear

k-linear restriction conjecture

If f_j is supported in U_j, $1 \leqslant j \leqslant k$, where $U_1, \dots, U_k \subset 2B^{n-1} \backslash B^{n-1}$ are transversal, i.e. $|G(U_1) \wedge \cdots \wedge G(U_k)| \gtrsim 1$, then

$$\left\| \prod_{j=1}^{k} |Ef_j|^{1/k} \right\|_{L^p(B_R)} \lesssim R^{\epsilon} \prod_{j=1}^{k} \|f_j\|_{L^2(2B^{n-1}\backslash B^{n-1})}^{1/k}$$

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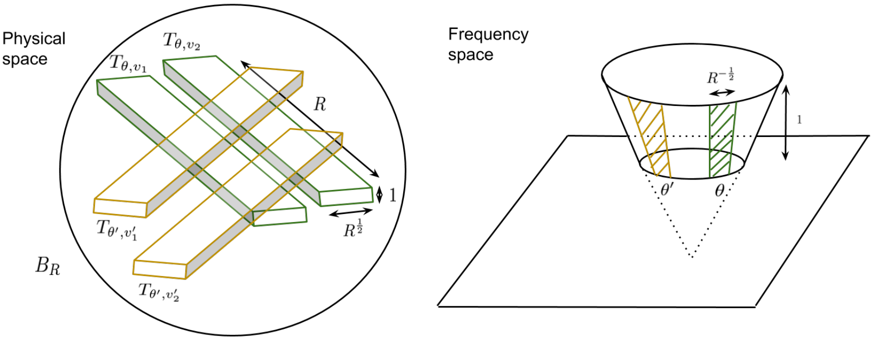
- Only known for $k = 2, n$ ([Wolff '01], [Bennett-Carbery-Tao '06]).
- k-broad estimate is a weaker substitute for this.
- Some schemes of using bilinear/multilinear restriction to prove linear restriction: [Wolff '01], [Tao '02], [Bourgain-Guth '11].

Wave packet decomposition



Wave packet decomposition: $f = \sum_{\theta,v} f_{\theta,v} + \mathrm{RapDec}(R)\|f\|_{L^2}$.

- supp $f_{\theta,v} \subset \theta$, $Ef_{\theta,v}$ is essentially supported and constant on $T_{\theta,v}$.
- Orthogonality: $\left\|\sum_{\theta,v} f_{\theta,v}\right\|_{L^2}^2 \approx \sum_{\theta,v} \|f_{\theta,v}\|_{L^2}^2$.

Polynomial partitioning

Theorem (Guth '16)

Fix constant D, there exists a polynomial P on $\mathbb{R}^n$ with $\deg(P) \leqslant D$ s.t. $Z(P)$ divides $\mathbb{R}^n \backslash Z(P)$ into disjoint union of $\sim D^n$ open sets O_i with equal measure $\mu_{Ef}(O_i)$, where μ_{Ef} is the measure determined by

$$\mu_{Ef}(B_{K^2}) = \min_{V_1,\ldots,V_A (k-1)\text{-subspace of } \mathbb{R}^n} \left(\max_{\tau \notin V_a, \forall a} \|Ef_\tau\|^p_{L^p(B_{K^2})} \right).$$

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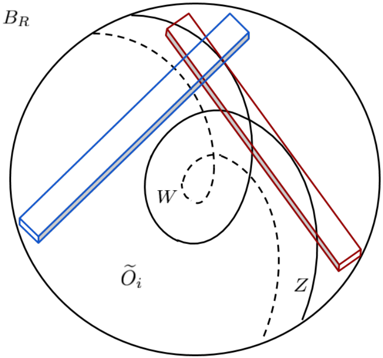
- Property: a straight line (not contained in $Z(P)$) can cross $Z(P)$ at most D times.
- Fatten up $Z(P)$ to wall $W = N_{R^{1/2}}(Z(P))$: make sure that each $T_{\theta,v}$ can intersect at most D cells $\widetilde{O}_i := O_i \backslash W$.

Three cases

Which part of B_R and f makes the most contribution to $\mu_{Ef}(B_R)$?



- $\sum_i \mu_{Ef}(\widetilde{O}_i)$ (Cellular case)
- $\mu_{Ef_{\text{trans}}}(W)$ where f_{trans} concentrates on the wave packets cutting cross Z (Transversal case)
- $\mu_{Ef_{\text{tang}}}(W)$ where f_{tang} concentrates on the wave packets tangential to Z (Tangential case)

Cellular case $\sum_i \mu_{Ef}(\widetilde{O}_i)$: induct on # of wave packets

Recall $\#\{\widetilde{O}_i\} \sim D^n$. $f_i := \sum_{T_{\theta,v} \cap \widetilde{O}_i \neq \varnothing} f_{\theta,v} \implies \mu_{Ef}(\widetilde{O}_i) = \mu_{Ef_i}(\widetilde{O}_i)$.

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- $\sum_i \mu_{Ef}(\widetilde{O}_i) \gtrsim \mu_{Ef}(B_R)$

$$\implies \text{most } i \text{ satisfies } \mu_{Ef}(\widetilde{O}_i) \gtrsim D^{-n} \mu_{Ef}(B_R). \qquad (3.1)$$

- Each $T_{\theta,v}$ intersects $\lesssim D$ different $\widetilde{O}_i$'s $\implies \sum_i \|f_i\|_{L^2}^2 \lesssim D\|f\|_{L^2}^2$

$$\implies \text{most } i \text{ satisfies } \|f_i\|_{L^2}^2 \lesssim D^{1-n}\|f\|_{L^2}^2. \qquad (3.2)$$

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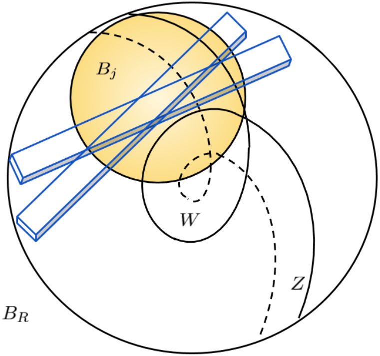
Fix i satisfying both, then

$$\mu_{Ef}(B_R) \overset{(3.1)}{\lesssim} D^n \mu_{Ef_i}(\tilde{O}_i) \overset{*}{\lesssim} D^n R^{\epsilon p}\|f_i\|_{L^2}^p \overset{(3.2)}{\lesssim} R^{\epsilon p} D^n D^{(1-n)p/2}\|f\|_{L^2}^p.$$

$*$: Induction hypothesis. Induction closes when $n + (1-n)p/2 \leqslant 0$.

Transversal case $\mu_{Ef_{\text{trans}}}(W)$: induct on R

Cover W with balls $\{B_j\}$ of radius $\rho = R^{1-\delta}$. $f_j := \sum_{T_{\theta,v} \cap B_j \neq \emptyset} f_{\theta,v}$.



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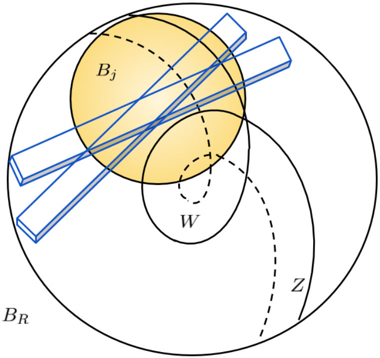
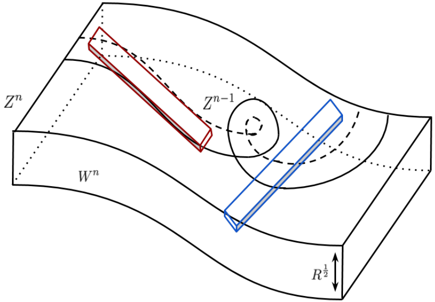


$$\mu_{Ef_{\text{trans}}}(W)$$
$$\leqslant \sum_j \mu_{Ef_j}(B_j \cap W)$$
$$\overset{*}{\lesssim} \sum_j \rho^{\epsilon p} \|f_j\|_{L^2}^p \leqslant \rho^{\epsilon p} \left(\sum_j \|f_j\|_{L^2}^2 \right)^{p/2}$$
$$\overset{**}{\lesssim} \rho^{\epsilon p} D^{np/2} \|f\|_{L^2}^p \leqslant R^{\epsilon p} \|f\|_{L^2}^p.$$

($*$: Induction hypothesis. $**$: Each $T_{\theta,v}$ can transversely cut through at most D^n B_j's.)

Tangential case $\mu_{Ef_{\mathrm{tang}}}(W)$: reduce dimension

This essentially becomes a restriction problem in $\mathbb{R}^{n-1}$, on which we perform another polynomial partitioning.



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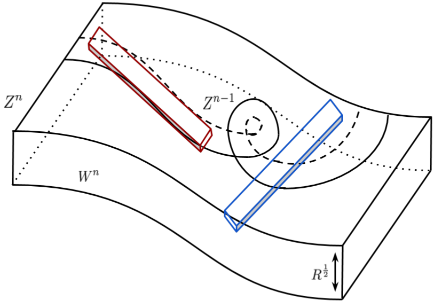

- Cellular case
- Tangential case: induct on dimension m of the variety Z. (Base case: $m = k - 1$, $\mu_{Ef}(B_{K^2}) = 0$, $\forall B_{K^2}$.)

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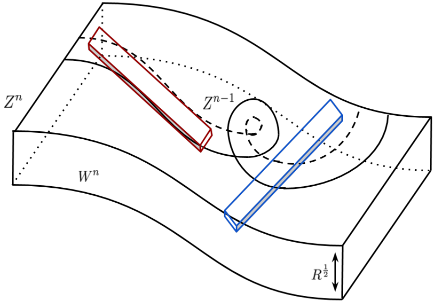

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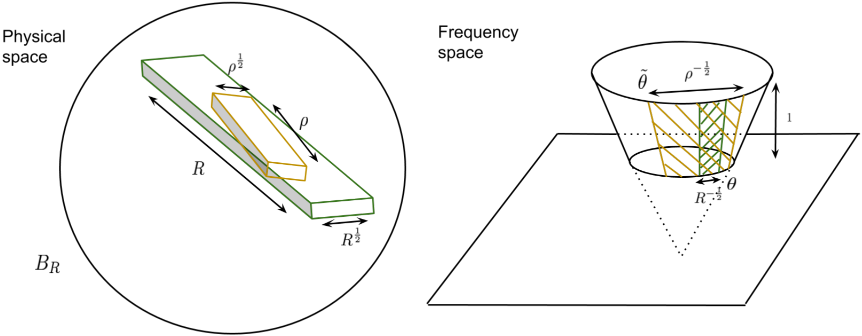
- Transversal case: the hard part!

The hard case: tangential in $\mathbb{R}^n$, transversal in $\mathbb{R}^{n-1}$



- Cover $W = N_{R^{1/2}}(Z^{n-1}) \subset \mathbb{R}^n$ with balls $\{B_j\}$ of radius $\rho < R$.
- Induct on R: for $\rho < R$, need new (smaller) wave packets $f = \sum_{\tilde{\theta},\tilde{v}} f_{\tilde{\theta},\tilde{v}} + \mathrm{RapDec}(\rho)\|f\|_{L^2}$ (strips $\tilde{\theta}$ of radius $\rho^{-1/2}$).
- $f_{\theta,v}$ concentrates on small wave packets that are roughly inside $T_{\theta,v}$ with angle $\lesssim \rho^{-1/2}$.

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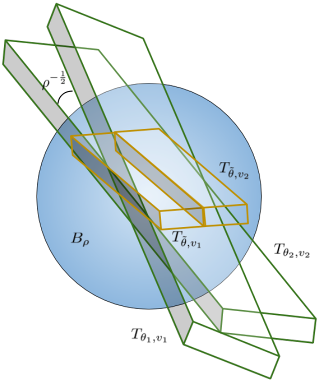

- The large and small wave packets are connected via medium tubes of sidelength $1 \times R^{1/2} \times \cdots \times R^{1/2} \times \rho$. (Locally, mini directions are roughly the same.)
- Induction hypothesis: if $\operatorname{supp} f \in N_{\rho^{1/2}}(Z^n)$ whose wave packets $T_{\tilde{\theta},\tilde{v}}$ are tangent to Z^n (Angle $< \rho^{-1/2}$, $T_{\tilde{\theta},\tilde{v}} \subset N_{\rho^{1/2}}(Z)$) and transversal to Z^{n-1}, then $\|Ef\|_{BL^p_{k,A}(B_j)} \lesssim \rho^\epsilon \|f\|_{L^2}$.
- Angle condition is satisfied, but distance condition is not.

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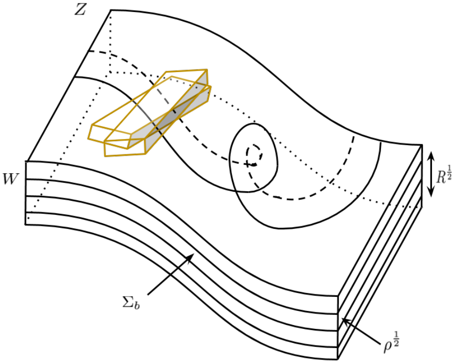
Solution: divide $W = N_{R^{1/2}}(Z)$ into layers $\{\Sigma_b\}$ of thickness $\rho^{1/2}$.



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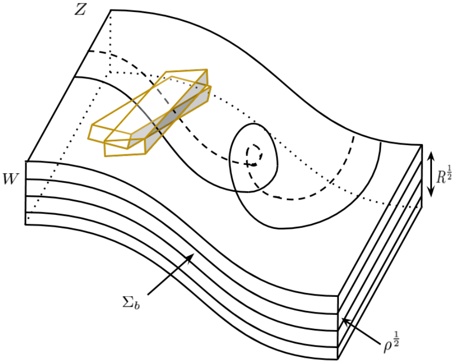
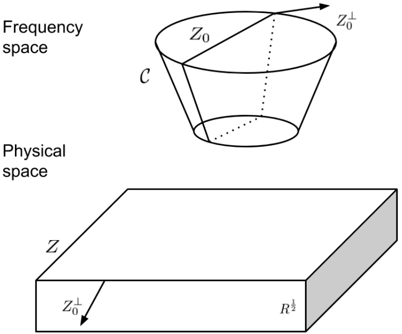


- Use transversal information to estimate contribution from each layer Σ_b (induct on R).
- Use tangential information to sum up layers: $\|f_{ess}\|_{L^2}$ is equidistributed across different layers.

Transverse equidistribution estimate: Z hyperplane

Lemma

Either a) supp $f \subset$ finitely many sectors τ (of radius K^{-1}) or b) Z is transversal to the orthogonal complement of Z_0.



- Case a): zero (by k-broad norm definition).
- Case b): supp f is contained in $N_{R^{-1/2}}(Z_0) \cap \mathcal{C} \implies Ef$ is locally constant along $Z_0^\perp$ for $R^{1/2}$, hence $\|Ef\|_{L^2}$ is equidistributed across the layers. Then by Plancherel, $\|f\|_{L^2}$ is also equidistributed.

Transverse equidistribution estimate: general case

- Run the argument locally inside each small ball $B = B_{R^{1/2}}$, which determines a tangent space V to Z.
- Divide the balls into two groups: whether V is in case a) or b). The wave packets that are covered by balls in group a) make zero contribution.
- Reduce to f_{ess} that concentrates on medium wave packets which intersect at least one ball B in group b).
- $\|Ef_{ess}\|_{L^2}$ is equidistributed across the layers. Same for $\|f_{ess}\|_{L^2}$.

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Thank you for your attention!