$T(1)$ and $T(b)$ Theorems on Product Spaces

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Classical SIO with standard kernel

Definition

$K : \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta \to \mathbb{C}$ is locally integrable away from the diagonal. It is called a *standard kernel*, if for some $0 < \delta \leq 1,\ C > 0$ we have

$$|K(x,t)| \leq C|x-t|^{-n}$$

$$|K(x,t) - K(x',t)| \leq C(|x-x'|)^{\delta}|x-t|^{-n-\delta}$$
$$|K(t,x) - K(t,x')| \leq C(|x-x'|)^{\delta}|x-t|^{-n-\delta}$$

whenever $|x-x'| \leq |x-t|/2$. $|K|$ denotes the best constant in both inequalities.

Space of δ-CZO

Definition

A continuous linear operator $T: C_0^\infty \to (C_0^\infty)'$ is called a *Calderón-Zygmund operator*, if it is bounded on $L^2(\mathbb{R}^n)$, and there is a standard kernel K such that for $\forall f, g \in C_0^\infty(\mathbb{R}^n)$ with disjoint supports,

$$\langle Tf, g\rangle = \int\int K(x,t)f(t)g(x)\,dtdx.$$

The set of all the δ-CZO is a Banach space (denoted by CZ) with norm $\|T\|_{CZ} := \|T\|_{L^2\to L^2} + |K|$.

Classical one-parameter $T(1)$ and $T(b)$ Theorems

Theorem ($T(1)$)

An operator $T : C_0^\infty \to (C_0^\infty)'$, associated with a standard kernel K, extends to a bounded operator on $L^2(\mathbb{R}^n)$ if and only if $T1, T^*1 \in BMO$ and T has the "weak boundedness property (WBP)".

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Theorem ($T(b)$)

Let b, b' be two accretive functions. An operator $T : bC_0^\infty \to (b'C_0^\infty)'$, associated with a standard kernel K, extends to a bounded operator on $L^2(\mathbb{R}^n)$ if and only if $Tb, T^*b' \in BMO$ and $M_{b'}TM_b$ has the WBP.

Example: Cauchy integral along Lipschitz curves

Let $y = A(x)$ be a Lipschitz function that defines a Lipschitz curve Γ in the complex plane. $A' \in L^\infty$. Define the Cauchy integral operator associated to Γ:

$$C_\Gamma f(x) = p.v. \int_{\mathbb{R}} \frac{f(y)}{(x-y) + i(A(x) - A(y))} \, dy.$$

Then, C_Γ is bounded on L^2.

(Check $C_\Gamma(1 + iA') = 0$ by Cauchy integral formula.)

Multi-parameter operators

Example 1 (Tensor product type)

For $f \in C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R})$, define the double Hilbert transform as

$$[(H_1 \otimes H_2)f](x_1, x_2) = \lim_{\substack{\epsilon_1 \to 0 \\ \epsilon_2 \to 0}} \int\int_{\substack{|x_1 - y_1| > \epsilon_1 \\ |x_2 - y_2| > \epsilon_2}} \frac{f(y_1, y_2)}{(x_1 - y_1)(x_2 - y_2)}\, dy_1 dy_2$$

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Example 2 (Non tensor product type)

For complex valued function $a \in L^\infty(\mathbb{R}^2)$ with $\|a\|_\infty < 1$, define the double Cauchy operator associated to the kernel K_a as

$$K_a(x, y) = \frac{1}{\prod_{i=1}^2 (x_i - y_i) + \int_{x_1}^{y_1} \int_{x_2}^{y_2} a(u_1, u_2) \, du}$$

Journé's class of bi-parameter SIO

Definition (Vector-valued Kernels)

Let B be a Banach space. A continuous function $K : \mathbb{R}^2 \setminus \Delta \to B$ is called a *B-δ-standard kernel*, if for some $0 < \delta \leq 1$, $C > 0$ we have

$$\|K(x,t)\|_B \leq C|x-t|^{-1}$$

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We will use the case $B = CZ$ in the following.

Definition (Bi-parameter SIO)

Let $T: C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R}) \to \left[C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R})\right]'$ be a continuous linear mapping. It is a δ-*SIO* on $\mathbb{R} \times \mathbb{R}$ if there exists a pair (K_1, K_2) of CZ-δ-standard kernels so that, $\forall f_1, f_2, g_1, g_2 \in C_0^\infty(\mathbb{R})$, with $\mathrm{supp} f_1 \cap \mathrm{supp} g_1 = \emptyset$,

$$\langle g_1 \otimes g_2, T(f_1 \otimes f_2)\rangle = \int\int g_1(x_1)\langle g_2, K_1(x_1, t_1) f_2\rangle f_1(t_1)\, dt_1 dx_1,$$

$$\langle g_2 \otimes g_1, T(f_2 \otimes f_1)\rangle = \int\int g_1(x_2)\langle g_2, K_2(x_2, t_2) f_2\rangle f_1(t_2)\, dt_2 dx_2.$$

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Partial adjoint operators:

$$\langle T_1(f_1 \otimes f_2), g_1 \otimes g_2\rangle = \langle T(g_1 \otimes f_2), f_1 \otimes g_2\rangle,$$

analogously for T_2. Note that $T_2 = T_1^*$.

We can check the double Hilbert transform is indeed in Journé's class.

- For fixed x_1, y_1, define $K_1(x_1, y_1)$ to be a constant multiple of the usual Hilbert transform:

$$K_1(x_1, y_1)h(x_2) = \frac{1}{x_1 - y_1} \lim_{\epsilon \to 0} \int_{|x_2 - y_2| > \epsilon} \frac{h(y_2)}{x_2 - y_2} \, dy_2.$$

It is CZ valued, with norm $\frac{C}{|x_1 - y_1|}$. Similarly for $K_2(x_2, y_2)$.

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- To check K_1 is associated to a standard kernel. It is easily seen that we have the size condition:

$$\|K_1(x_1, y_1)\|_{CZ} \leq \frac{C}{|x_1 - y_1|}.$$

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- For Hölder conditions, note that $\|\nabla K(x_1, y_1)\|_{CZ} \leq \frac{C}{|x_1 - y_1|^2}$.

BMO in the product spaces

- Standard dyadic grid:

$$\mathcal{D}^0 := \{2^{-k}([0,1)^d + m) \, : \, k \in \mathbb{Z}, m \in \mathbb{Z}^d\}$$

- Shifted dyadic grid: Let $\omega = (\omega_j)_{j\in\mathbb{Z}} \in (\{0,1\}^d)^{\mathbb{Z}}$ and

$$I \dot{+} \omega := I + \sum_{j:2^{-j}<l(I)} 2^{-j}\omega_j.$$

Then

$$\mathcal{D}^\omega := \{I \dot{+} \omega \, : \, I \in \mathcal{D}^0\}.$$

Dyadic double square function

For any given dyadic grids $\mathcal{D}_n, \mathcal{D}_m$ in $\mathbb{R}^n, \mathbb{R}^m$, respectively, define

$$S^2_{\mathcal{D}_n\mathcal{D}_m}(f) = \sum_{K\in\mathcal{D}_n} \sum_{V\in\mathcal{D}_m} |\langle f, h_K \otimes u_V\rangle|^2 \frac{\chi_K \otimes \chi_V}{|K||V|}.$$

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- We say a function $f \in H^1_{\mathcal{D}_n\mathcal{D}_m}(\mathbb{R}^n \times \mathbb{R}^m)$ if $S_{\mathcal{D}_n\mathcal{D}_m} f \in L^1(\mathbb{R}^n \times \mathbb{R}^m)$. Let $\|f\|_{H^1_{\mathcal{D}_n\mathcal{D}_m}} = \|Sf\|_{L^1}$.
- $BMO_{\mathcal{D}_n\mathcal{D}_m}$ is defined as the dual of $H^1_{\mathcal{D}_n\mathcal{D}_m}$.

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- $BMO_{\mathcal{D}_n\mathcal{D}_m}$ is defined as the dual of $H^1_{\mathcal{D}_n\mathcal{D}_m}$.

Definition

f is in *product BMO* iff f is in product dyadic BMO uniformly with respect to every shifted dyadic grid.

Bi-parameter $T(1)$ Theorem 1

Theorem (J. Journé)

Let T be a δ-SIO on $\mathbb{R} \times \mathbb{R}$ satisfying the WBP and $T(1), T^*(1), T_1(1), T_1^*(1) \in BMO_{prod}(\mathbb{R} \times \mathbb{R})$. Then T extends boundedly on $L^2(\mathbb{R} \times \mathbb{R})$.

Bi-parameter $T(1)$ Theorem 1

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Simplest case: $T(1) = T^*(1) = T_1(1) = T_1^*(1) = 0$

View T as a classical vector valued SIO acting on $C_0^\infty(\mathbb{R}) \otimes L^2(\mathbb{R}, dx_2)$, and use the hilbertian version of the classical T(1) argument.

Pott-Villarroya's class of bi-parameter SIO

Novelty:

- Drop the a priori partial boundedness assumption.
- Replace the vector-valued assumptions by an enlarged set of "mixed conditions" combining more size and Hölder conditions together.

P-V's class of bi-parameter SIO (reformulated by Martikainen)

For $f = f_1 \otimes f_2, g = g_1 \otimes g_2$ with $f_1, g_1 : \mathbb{R}^n \to \mathbb{C}, f_2, g_2 : \mathbb{R}^m \to \mathbb{C}$, and $\mathrm{spt} f_1 \cap \mathrm{spt} g_1 = \mathrm{spt} f_2 \cap \mathrm{spt} g_2 = \emptyset$, we have

$$\langle Tf, g \rangle = \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^{n+m}} K(x, y) f(y) g(x) \, dxdy.$$

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$$\langle Tf, g \rangle = \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^{n+m}} K(x, y) f(y) g(x) \, dxdy.$$

(1) Size condition: $|K(x, y)| \leq C \frac{1}{|x_1 - y_1|^n} \frac{1}{|x_2 - y_2|^m}$.

(2) Hölder conditions:

1. $|K(x,y) - K(x,(y_1,y_2')) - K(x,(y_1',y_2)) + K(x,y')| \leq C\frac{|y_1-y_1'|^\delta}{|x_1-y_1|^{n+\delta}}\frac{|y_2-y_2'|^\delta}{|x_2-y_2|^{m+\delta}}$ whenever $|y_1 - y_1'| \leq |x_1 - y_1|/2$ and $|y_2 - y_2'| \leq |x_2 - y_2|/2$,
2. $|K(x,y) - K((x_1,x_2'),y) - K((x_1',x_2),y) + K(x',y)| \leq C\frac{|x_1-x_1'|^\delta}{|x_1-y_1|^{n+\delta}}\frac{|x_2-x_2'|^\delta}{|x_2-y_2|^{m+\delta}}$ whenever $|x_1 - x_1'| \leq |x_1 - y_1|/2$ and $|x_2 - x_2'| \leq |x_2 - y_2|/2$,
3. $|K(x,y) - K((x_1,x_2'),y) - K(x,(y_1',y_2)) + K((x_1,x_2'),(y_1',y_2))| \leq C\frac{|y_1-y_1'|^\delta}{|x_1-y_1|^{n+\delta}}\frac{|x_2-x_2'|^\delta}{|x_2-y_2|^{m+\delta}}$ whenever $|y_1 - y_1'| \leq |x_1 - y_1|/2$ and $|x_2 - x_2'| \leq |x_2 - y_2|/2$,
4. $|K(x,y) - K(x,(y_1,y_2')) - K((x_1',x_2),y) + K((x_1',x_2),(y_1,y_2'))| \leq C\frac{|x_1-x_1'|^\delta}{|x_1-y_1|^{n+\delta}}\frac{|y_2-y_2'|^\delta}{|x_2-y_2|^{m+\delta}}$ whenever $|x_1 - x_1'| \leq |x_1 - y_1|/2$ and $|y_2 - y_2'| \leq |x_2 - y_2|/2$.

(3) Mixed Hölder and size conditions:

1. $|K(x,y) - K((x_1',x_2),y)| \le C \frac{|x_1-x_1'|^\delta}{|x_1-y_1|^{n+\delta}} \frac{1}{|x_2-y_2|^m}$
   whenever $|x_1 - x_1'| \le |x_1 - y_1|/2$,
2. $|K(x,y) - K(x,(y_1',y_2))| \le C \frac{|y_1-y_1'|^\delta}{|x_1-y_1|^{n+\delta}} \frac{1}{|x_2-y_2|^m}$
   whenever $|y_1 - y_1'| \le |x_1 - y_1|/2$,
3. $|K(x,y) - K((x_1,x_2'),y)| \le C \frac{1}{|x_1-y_1|^n} \frac{|x_2-x_2'|^\delta}{|x_2-y_2|^{m+\delta}}$
   whenever $|x_2 - x_2'| \le |x_2 - y_2|/2$,
4. $|K(x,y) - K(x,(y_1,y_2'))| \le C \frac{1}{|x_1-y_1|^n} \frac{|y_2-y_2'|^\delta}{|x_2-y_2|^{m+\delta}}$
   whenever $|y_2 - y_2'| \le |x_2 - y_2|/2$.

(4) Partial CZ structure:
If $f = f_1 \otimes f_2$ and $g = g_1 \otimes g_2$ with $\mathrm{spt} f_1 \cap \mathrm{spt} g_1 = \emptyset$. Then assume

$$\langle Tf, g\rangle = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K_{f_2,g_2}(x_1, y_1) f_1(y_1) g_1(x_1)\, dx_1 dy_1.$$

The kernel $K_{f_2,g_2} : (\mathbb{R}^n \times \mathbb{R}^n) \setminus \{x_1 = y_1\}$ satisfies:

- $|K_{f_2,g_2}(x_1, y_1)| \leq C(f_2, g_2)\frac{1}{|x_1 - y_1|^n}$
- $|K_{f_2,g_2}(x_1, y_1) - K_{f_2,g_2}(x_1', y_1)| \leq C(f_2, g_2)\frac{|x_1 - x_1'|^\delta}{|x_1 - y_1|^{n+\delta}}$ whenever $|x_1 - x_1'| \leq |x_1 - y_1|/2$,
- $|K_{f_2,g_2}(x_1, y_1) - K_{f_2,g_2}(x_1, y_1')| \leq C(f_2, g_2)\frac{|y_1 - y_1'|^\delta}{|x_1 - y_1|^{n+\delta}}$ whenever $|y_1 - y_1'| \leq |x_1 - y_1|/2$.

- Control for $C(f_2, g_2)$ in the diagonal:
  For any cube $V \subset \mathbb{R}^m$,
  $C(\chi_V, \chi_V) + C(\chi_V, u_V) + C(u_V, \chi_V) \leq C|V|$, whenever u_V is V-adapted with zero-mean (i.e. spt$u_V \subset V, |u_V| \leq 1, \int u_V = 0$).

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And we also assume the analogous representation and properties with a kernel K_{f_1, g_1} in the case spt$f_2 \cap$ spt$g_2 = \emptyset$.

Equivalence of the two classes

- A very recent result of A. Grau De La Herrán shows that under the additional condition that T is bounded on L^2, the classes of bi-parameter SIO defined by Journé and P-V are equivalent.
- This implies in particular that for a SIO T in P-V's class to be bounded on L^2, conditions $T(1), T^*(1) \in BMO$ are necessary.

Boundedness and cancellation assumptions

- $T1, T^*1, T_1(1), T_1^*(1) \in BMO(\mathbb{R}^n \times \mathbb{R}^m)$.

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- (Dyadic) WBP: $|\langle T(\chi_K \otimes \chi_V), \chi_K \otimes \chi_V \rangle| \leq C|K||V|$ for every cube $K \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$.

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- Diagonal BMO conditions: for any cube $K \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$, and every zero-mean functions a_K and b_V which are K and V adapted respectively:
  1. $|\langle T(a_K \otimes \chi_V), \chi_K \otimes \chi_V\rangle| \le C|K||V|$
  2. $|\langle T(\chi_K \otimes \chi_V), a_K \otimes \chi_V\rangle| \le C|K||V|$
  3. $|\langle T(\chi_K \otimes b_V), \chi_K \otimes \chi_V\rangle| \le C|K||V|$
  4. $|\langle T(\chi_K \otimes \chi_V), \chi_K \otimes b_V\rangle| \le C|K||V|$

Bi-parameter $T(1)$ Theorem 2

Theorem (H. Martikainen)

For a bi-parameter SIO T of P-V's class satisfying the boundedness and cancellation assumptions, there holds for some bi-parameter shifts $S^{i_1 i_2 j_1 j_2}_{\mathcal{D}_n \mathcal{D}_m}$ that

$$\langle Tf, g\rangle = C\mathbb{E}_{\omega_n}\mathbb{E}_{\omega_m} \sum_{i_1,i_2,j_1,j_2 \geq 0} 2^{-max(i_1,i_2)\frac{\delta}{2}} 2^{-max(j_1,j_2)\frac{\delta}{2}} \langle S^{i_1 i_2 j_1 j_2}_{\mathcal{D}_n \mathcal{D}_m} f, g\rangle.$$

Corollary

A bi-parameter singular integral T as defined above is L^2 bounded.

Bi-parameter $T(b)$ Theorem

Definition

A function $b \in L^\infty(\mathbb{R}^n \times \mathbb{R}^m)$ is called pseudo-accretive if there is a constant C such that for any rectangle R in $\mathbb{R}^n \times \mathbb{R}^m$ with sides parallel to axes, $\frac{1}{|R|}|\int_R b| > C$.

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Theorem (Y. Ou)

Let $b = b_1 \otimes b_2, b' = b_1' \otimes b_2'$ be pseudo-accretive. For a bi-parameter SIO T of a class similar to P-V's, satisfying $Tb, T^*b', T_1(b_1' \otimes b_2), T_1^*(b_1 \otimes b_2') \in BMO(\mathbb{R}^n \times \mathbb{R}^m)$, and $M_{b'}TM_b$ has the WBP, then T is bounded on $L^2(\mathbb{R}^n \times \mathbb{R}^m)$.

Assumptions on the SIO for $T(b)$ Theorem

(1) Full C-Z structure:
If $f = f_1 \otimes f_2$ and $g = g_1 \otimes g_2$ with $f_1, g_1 \in C_0^\infty(\mathbb{R}^n)$, $f_2, g_2 \in C_0^\infty(\mathbb{R}^m)$, $\text{spt} f_1 \cap \text{spt} g_1 = \emptyset$ and $\text{spt} f_2 \cap \text{spt} g_2 = \emptyset$, then

$$\langle M_{b'} T M_b f, g \rangle = \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^{n+m}} K(x,y) f(y) g(x) b(y) b'(x) \, dxdy.$$

The kernel K satisfies all the size, Hölder and mixed size-Hölder conditions.

(2) Partial C-Z structure:
If $f = f_1 \otimes f_2, g = g_1 \otimes g_2$ and spt$f_1 \cap$ spt$g_1 = \emptyset$, then

$$\langle M_{b'} T M_b f, g\rangle = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K_{f_2,g_2}(x_1,y_1) f_1(y_1) g_1(x_1) b_1(y_1) b_1'(x_1)\, dx_1 dy_1 .$$

The partial kernel K_{f_2,g_2} satisfies all the partial size and Hölder conditions with constant $C(f_2, g_2)$ satisfying

$$C(\chi_V, \chi_V) + C(\chi_V, u_V b_2'^{-1}) + C(u_V b_2^{-1}, \chi_V) \leq C|V|.$$

We also assume the symmetric partial kernel representation and conditions on kernel K_{f_1,g_1} for the other variable.

(3) Boundedness and cancellation assumptions:

- WBP: $|\langle M_{b'} T M_b(\chi_K \otimes \chi_V), \chi_K \otimes \chi_V \rangle| \leq C|K||V|$.
- BMO: Let $d = b_1 \otimes b_2'$, $d' = b_1' \otimes b_2$, assume Tb, $T^* b'$, $T_1 d'$, $T_1^* d \in BMO(\mathbb{R}^n \times \mathbb{R}^m)$.
- Diagonal BMO: Let a_K, b_V be any K, V adapted zero-mean functions,
  - $|\langle M_{b'} T M_b(a_K b_1^{-1} \otimes \chi_V), \chi_K \otimes \chi_V \rangle| \leq C|K||V|$
  - $|\langle M_{b'} T M_b(\chi_K \otimes \chi_V), a_K b_1'^{-1} \otimes \chi_V \rangle| \leq C|K||V|$
  - $|\langle M_{b'} T M_b(\chi_K \otimes b_V b_2^{-1}), \chi_K \otimes \chi_V \rangle| \leq C|K||V|$
  - $|\langle M_{b'} T M_b(\chi_K \otimes \chi_V), \chi_K \otimes b_V b_2'^{-1} \rangle| \leq C|K||V|$

Difficulties from $T(1)$ to $T(b)$

- Define $T(b)$. (In the case b is a tensor product)
- Develop estimates of bi-parameter b-adapted paraproducts.
- Generalize the argument for dyadic shifts to deal with b-adapted martingale differences directly instead of Haar basis.

Bi-parameter *b*-adapted martingale differences

For each $p \in \mathbb{Z}$, let $\mathcal{D}_p^n$ be the collection of cubes of side length 2^{-p} in $\mathcal{D}^n$, define

$$E_p^{b_1} f = \sum_{I \in \mathcal{D}_p^n} \frac{\int_I f b_1}{\int_I b_1} \chi_I, \quad E_I^{b_1} f = \chi_I E_p^{b_1} f.$$

Similarly for the other variable. Then

$$E_{p,q}^{b} = E_p^{b_1} E_q^{b_2} = E_q^{b_2} E_p^{b_1}.$$

Let $\Delta_p^{b_1} = E_{p+1}^{b_1} - E_p^{b_1}$, $\Delta_I^{b_1} = \chi_I \Delta_p^{b_1}$ for each $I \in \mathcal{D}_p^n$, similarly for the other variable. The *b*-adapted double martingale difference is defined as

$$\Delta_{p,q}^{b} = \Delta_p^{b_1} \Delta_q^{b_2} = \Delta_q^{b_2} \Delta_p^{b_1}.$$

Properties

1. $\Delta^b_{I\times J} f$ is supported on the dyadic rectangle $I \times J$, and is a constant on each of its children;
2. $\int b_1 \Delta^b_{p,q} f \, dx_1 = \int b_2 \Delta^b_{p,q} f \, dx_2 = 0$;
3. $\Delta^b_{p,q}\Delta^b_{k,l} = 0$ unless $p = k$, $q = l$, and in this case it equals $\Delta^b_{p,q}$;
4. If $f \in L^2(\mathbb{R}^n \times \mathbb{R}^m)$, then $f = \sum_{p,q} \Delta^b_{p,q} f$ with convergence in L^2, and
$$\|f\|^2_{L^2} \lesssim \sum_{p,q} \|\Delta^b_{p,q} f\|^2_{L^2} \lesssim \|f\|^2_{L^2}.$$
5. $M_b \Delta^b_{p,q} = \Delta^{b*}_{p,q} M_b$

Definition of $T(b)$

Let A be the subspace of $H^1_d(\mathbb{R}^n \times \mathbb{R}^m)$ containing all the finite combinations of $b'\Delta_I^{b_1'}\Delta_J^{b_2'}f$. Then, "$Tb \in BMO$" means "$Tb$ is a bounded functional on A".
To define

$$\langle b'\Delta_I^{b_1'}\Delta_J^{b_2'}f, Tb\rangle,$$

split into

$$\langle b'\Delta_I^{b_1'}\Delta_J^{b_2'}f, T(b\chi_{3I}\otimes\chi_{3J})\rangle + \langle b'\Delta_I^{b_1'}\Delta_J^{b_2'}f, T(b\chi_{3I}\otimes\chi_{(3J)^c})\rangle$$
$$+ \langle b'\Delta_I^{b_1'}\Delta_J^{b_2'}f, T(b\chi_{(3I)^c}\otimes\chi_{3J})\rangle + \langle b'\Delta_I^{b_1'}\Delta_J^{b_2'}f, T(b\chi_{(3I)^c}\otimes\chi_{(3J)^c})\rangle.$$

Tools: *b*-adapted bi-parameter paraproduct

Definition

For $a \in BMO(\mathbb{R}^n \times \mathbb{R}^m)$, operator $\pi_a^{b',b}$ is called *full paraproduct*, defined as

$$\pi_a^{b',b}(f) = \sum_{K \in \mathcal{D}^n, V \in \mathcal{D}^m} \langle f \rangle_{K \times V}^{b'} M_b \Delta_K^{b_1} \Delta_V^{b_2} a.$$

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Proposition

Full paraproducts are bounded operators on $L^2(\mathbb{R}^n \times \mathbb{R}^m)$. *Specifically,*

$$\|\pi_a^{b',b}(f)\|_{L^2(\mathbb{R}^n \times \mathbb{R}^m)} \lesssim \|a\|_{BMO(\mathbb{R}^n \times \mathbb{R}^m)} \|f\|_{L^2(\mathbb{R}^n \times \mathbb{R}^m)}.$$

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In the definition, if f is averaged only on one variable but has a martingale difference attached to the other one, what obtained is called a *mixed paraproduct*, which satisfies a similar L^2 estimate.

Key steps in the proof of the proposition:

1. Define b-adapted square function and maximal function:

$$S_b f(x) = \Big(\sum_{p,q\in\mathbb{Z}} |\Delta_p^{b_1}\Delta_q^{b_2} f(x)|^2\Big)^{1/2} = \Big(\sum_{I\in\mathcal{D}^n, J\in\mathcal{D}^m} |\Delta_I^{b_1}\Delta_J^{b_2} f(x)|^2\Big)^{1/2},$$

$$f_b^*(x) = \sup_{p,q\in\mathbb{Z}} |E_p^{b_1}E_q^{b_2} f(x)| = \sup_{I\in\mathcal{D}^n, J\in\mathcal{D}^m} |E_I^{b_1}E_J^{b_2} f(x)|.$$

2. Observe $\|f\|_{H_b^1} \approx \|f_b^*\|_{L^1}$, where $f \in H_b^1$ iff $fb \in H^1$.
3. Prove $\|f_b^*\|_{L^1} \lesssim \|S_b f\|_{L^1}$.

Demonstrate Step 3

Define $K_b^1(\mathbb{R}^n \times \mathbb{R}^m) = \{f : \ S_b f \in L^1(\mathbb{R}^n \times \mathbb{R}^m)\}$.

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Theorem (Atomic Decomposition)

Given $f \in K_b^1$, there exists a sequence of atoms a^n and a sequence of scalars λ_n such that (1) $f = \sum_n \lambda_n a^n$, a.e. (2) $\sum_n |\lambda_n| \lesssim \|f\|_{K_b^1}$.

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Proposition

If a is an atom, then $a \in C \cdot B$, where B is the unit ball in H_b^1 or K_b^1, and C is a universal constant independent of a.

Tools: Control lemma imitating "dyadic shifts"

Lemma

For fixed $i_1, i_2, j_1, j_2 \in \mathbb{N}$ *and any* $f \in L^2(\mathbb{R}^n \times \mathbb{R}^m)$, $g \in L^2(\mathbb{R}^n \times \mathbb{R}^m)$,

$$\sum_{\substack{K \in \mathcal{D}^n \\ V \in \mathcal{D}^m}} \sum_{I_1, I_2 \subset K}^{(i_1, i_2)} \sum_{J_1, J_2 \subset V}^{(j_1, j_2)} \frac{|I_1|^{1/2} |I_2|^{1/2}}{|K|} \frac{|J_1|^{1/2} |J_2|^{1/2}}{|V|} \|\Delta_{I_1}^{b_1} \Delta_{J_1}^{b_2} f\|_{L^2} \|\Delta_{I_2}^{b_1'} \Delta_{J_2}^{b_2'} g\|_{L^2}$$

$$\lesssim \|f\|_{L^2} \|g\|_{L^2},$$

where the constant doesn't depend on i_1, i_2, j_1, j_2.

Sketch of the Proof of $T(b)$ Theorem

1. Averaging formula

$$\langle M_{b'} T M_b f, g \rangle =$$
$$\frac{1}{\pi^n_{\text{good}} \pi^m_{\text{good}}} \mathbb{E}_{\omega^n} \mathbb{E}_{\omega^m} \sum_{\substack{I_1, I_2 \in \mathcal{D}^n \\ J_1, J_2 \in \mathcal{D}^m}} \chi_{\text{good}}(\text{sm}(I_1, I_2)) \chi_{\text{good}}(\text{sm}(J_1, J_2)) \cdot$$
$$\langle M_{b'} T M_b \Delta^{b_1}_{I_1} \Delta^{b_2}_{J_1} f, \Delta^{b'_1}_{I_2} \Delta^{b'_2}_{J_2} g \rangle.$$

2. Case by case manipulation: relative positions of cubes

Good and bad cubes

Definition

A cube I is called bad if there exists $\tilde{I} \in \mathcal{D}_n$ so that $l(\tilde{I}) \geq 2^r l(I)$ and $d(I, \partial\tilde{I}) \leq 2l(I)^{\gamma_n} l(\tilde{I})^{1-\gamma_n}$, where $\gamma_n = \delta/(2n + 2\delta)$.

- For $I \in \mathcal{D}^0$, the position and badness of $I \dotplus \omega$ are independent random variables.
- The probability of a particular cube $I \dotplus \omega$ being bad is equal for all $I \in \mathcal{D}^0$:
$$\mathbb{P}_\omega(I \dotplus \omega \text{bad}) = \pi_{\text{bad}} = \pi_{\text{bad}}(r, n, \delta)$$
- Key lemma: $\pi_{\text{bad}} < 1$ if $r = r(n, \delta)$ is chosen large enough.

Thank you!